

Column representation of words in cellular automata

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Definition

A one-dimensional *cellular automaton* (CA) with memory d is a dynamical system $\left((\mathcal{A}^d)^{\mathbb{Z}}, F \right)$, where $\mathcal{A}$ is a finite set, $d \geq 1$, and where the map $F : (\mathcal{A}^d)^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ is defined by a local rule which acts uniformly and synchronously on the configuration space.

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Precisely, there exists an integer $r \geq 0$ called the *radius* of the CA, and a *local rule* $\phi : \left(\mathcal{A}^d\right)^{2r+1} \rightarrow \mathcal{A}$ such that $\forall x \in \mathcal{A}^{\mathbb{Z}}, \forall k \in \mathbb{Z}, F(x)_m = \phi((x_{m+i})_{-r \leq i \leq r})$.

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Definition

If $F : (\mathcal{A}^d)^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ is a cellular automaton with memory d , a *spacetime diagram* for F with initial conditions $R_0, \dots, R_{d-1}$ is a sequence $(R_n)_{n \geq 0}$ where we inductively define $R_n = F(R_{n-d}, \dots, R_{n-1})$ for $n \geq d$.

Cellular automata of the sum mod 2

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With $r = 1$, $d = 1$, $\mathcal{A} = \{0, 1\}$ and $\phi(a, b, c) = a + c \bmod 2$.

Initial condition $R(0) = 1$ and $R(m) = 0$ for $m \neq 0$.

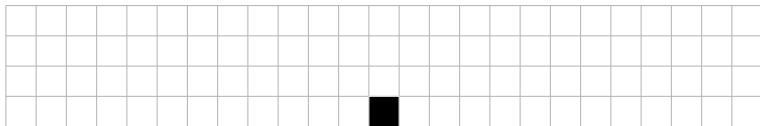


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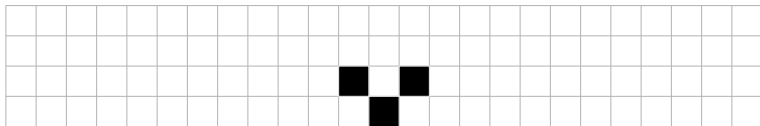


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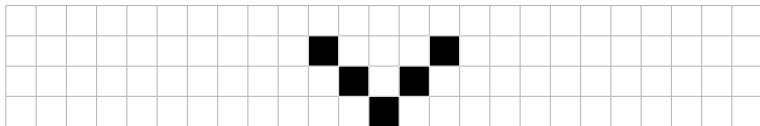


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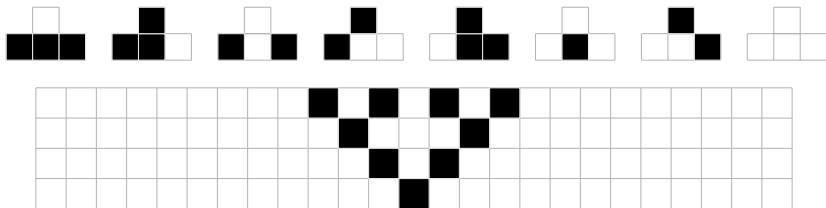


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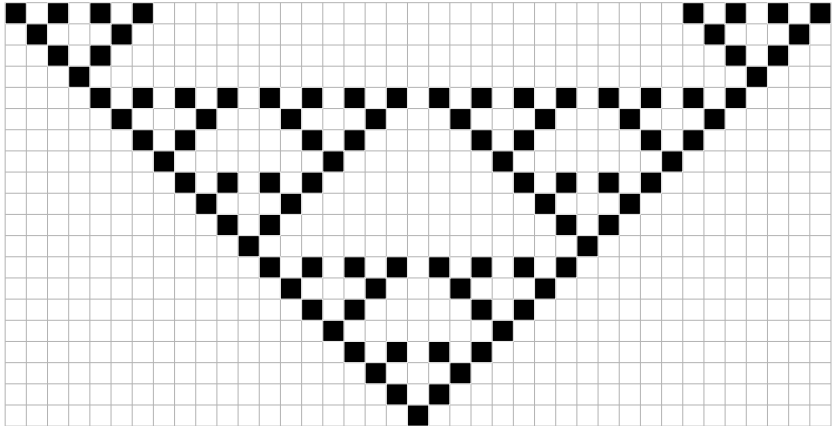
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Cellular automata of the sum mod 2



Definition

Let $\mathcal{A}$ be a finite set containing an element denoted by 0, and let $F : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ be a CA.

We say that F is *0-quiescent* if $F(0^{\mathbb{Z}}) = 0^{\mathbb{Z}}$. A configuration $x \in \mathcal{A}^{\mathbb{Z}}$ is said to be *finite* if the set $\{k \in \mathbb{Z} : x_k \neq 0\}$ is finite.

Cellular Automata

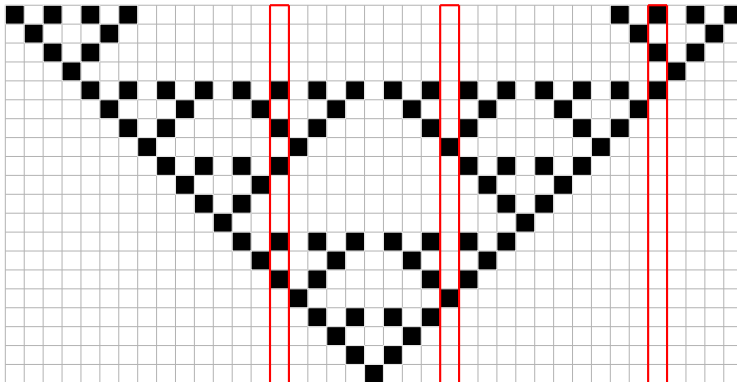
We consider the set

$$\mathcal{S} = \left\{ (F^n(x)_0)_{n \geq 0} \in \mathcal{A}^{\mathbb{N}} : F \text{ is a 0-quiescent CA on } \mathcal{A}^{\mathbb{Z}} \text{ and } x \text{ is finite} \right\}$$

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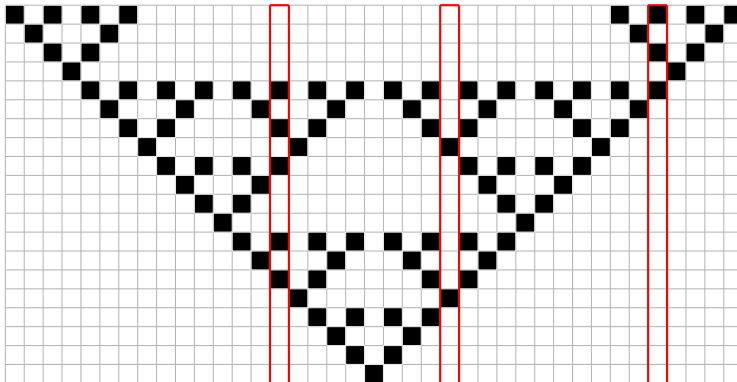
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Which infinite sequences belong to $\mathcal{S}$?

Fischer's constructible functions

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- Dolce, Tahay (2022): Constructions of Sturmian words with quadratic slope.

Definition

A deterministic finite automaton with output (DFAO) is a 6-tuple $(Q, \Sigma_k, \delta, s_0, \mathcal{A}, \omega)$ where Q is a finite set of *states*, $\Sigma_k = \{0, 1, \dots, k-1\}$, $s_0 \in Q$ is the initial state, $\mathcal{A}$ is a finite alphabet, $\omega : Q \rightarrow \mathcal{A}$ is the output function, and $\delta : Q \times \Sigma_k \rightarrow Q$ is the transition function.

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Definition

A sequence $(u_n)_{n \geq 0}$ of elements in $\mathcal{A}$ is k -automatic if there is a DFAO $(Q, \Sigma_k, \delta, s_0, \mathcal{A}, \omega)$ such that $u_n = \omega(\delta(s_0, (n)_k))$ for all $n \geq 0$.

Example

The Rudin-Shapiro sequence $(r_n)_{n \geq 0} = 0, 0, 0, 1, 0, 0, 1, 0, \dots$ is defined for all $n \in \mathbb{N}$ by:

$$r_n = (\text{number of } 11 \text{ in the binary representation of } n) \bmod 2$$

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For $n = 55 = (110111)_2$ we have $r_{55} = 3 \bmod 2 = 1$

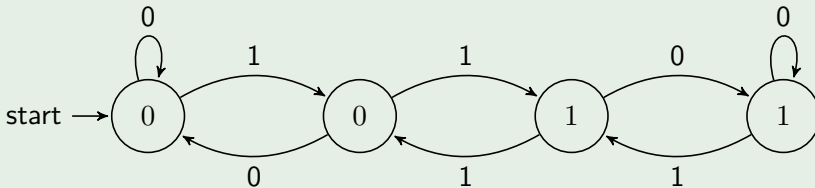
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It is generated by the following automaton:



Definition

Let q a power of a prime number. We say that the cellular automaton $F : (\mathbb{F}_q^d)^{\mathbb{Z}} \rightarrow \mathbb{F}_q^{\mathbb{Z}}$ with memory d is *linear* if F is an $\mathbb{F}_q$ -linear map.

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Theorem (Curtis-Hedlund-Lyndon)

A cellular automaton with memory d is linear if and only if there exists coefficients $f_{j,i} \in \mathbb{F}_q$, for $-r \leq j \leq r$ and $0 \leq i \leq d-1$ such that

$$F(R_0, \dots, R_{d-1})_m = \sum_{i=0}^{d-1} \sum_{j=-r}^r f_{j,i} R_i(m+j)$$

for all $R_0, \dots, R_{d-1} \in \mathbb{F}_q^{\mathbb{Z}}$ and $m \in \mathbb{Z}$.

Theorem (Litow, Dumas, 1993)

Let p be a prime number. Every column of a linear CA over $\mathbb{F}_p$ (with finite initial configuration) is p -automatic.

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Theorem (Rowland, Yassawi, 2015)

Let p be a prime number and q be a power of p . A sequence of elements in $\mathbb{F}_q$ is p -automatic if and only if it is a column of a spacetime diagram of a linear cellular automaton with memory over $\mathbb{F}_q$ whose initial conditions are eventually periodic in both directions.

Characterization of q -automatic sequences

Remark

Let p be a prime number. Given that for all $n \geq 1$ a sequence is p -automatic if and only if it is p^n -automatic, the result of Rowland and Yassawi extends to all q -automatic sequences where q is a power of p .

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The proof of Rowland and Yassawi is constructive. They give an explicit method to construct a cellular automaton where a given p -automatic sequence occurs in a column.

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Theorem (Christol, 1980)

Let q be a power of a prime number. Let $(u_n)_{n \geq 0}$ be a sequence of elements in $\mathbb{F}_q$. Then, $(u_n)_{n \geq 0}$ is a q -automatic sequence, if and only if $F(t) = \sum_{n \geq 0} u_n t^n$ is algebraic over $\mathbb{F}_q(t)$.

Principle of the method

We encode the spacetime diagram of a cellular automaton by the series $\sum_{n \geq 0} \sum_{m \in \mathbb{Z}} a_{n,m} t^n x^m$, where $a_{n,m}$ represents the cell on row $n \in \mathbb{N}$ and column $m \in \mathbb{Z}$.

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Let $(u_n)_{n \geq 0}$ a p -automatic sequence of elements in $\mathbb{F}_q$ and let

$F(t) = \sum_{n \geq 0} u_n t^n$ its generating function.

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encodes a spacetime diagram of a cellular automaton where the sequence $(u_n)_{n \geq 0}$ occurs in the column -2 , and where for all $n \in \mathbb{N}$ and all $m \in \mathbb{Z}$ the coefficient x^m in $R_n(x)$ represents the cell on row n and column m .

Construction of Rudin–Shapiro sequence

Example

Let $(r_n)_{n \geq 0}$ be the Rudin–Shapiro sequence. Let $F(t) = \sum_{n \geq 0} r_n t^n$.

We can show that, for the polynomial

$$P(t, x) = t^2 + (t^4 + 1)x + (t + t^2 + t^5 + t^6)x^2.$$

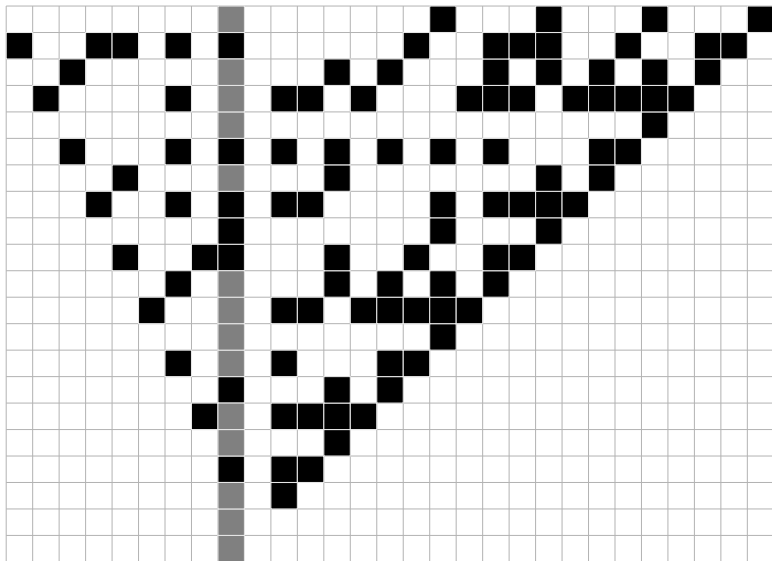
we have $P(t, F(t)) = 0$

With this polynomial we can define a cellular automaton, in which the Rudin–Shapiro sequence occurs in the column -2 , with

$$\begin{aligned} R_n(x) = & xR_{n-1}(x) + \left(\frac{1}{x} + x\right) R_{n-2}(x) + R_{n-4}(x) \\ & + xR_{n-5}(x) + xR_{n-6}(x), \end{aligned}$$

for all $n \geq 7$.

Construction of Rudin–Shapiro sequence



Definition

Let p be a prime number. The sequence $(a(n))_{n \geq 0}$ defined on the alphabet $\{0, 1, \dots, p-1\}$ by

$$a(pn + j) = (a(n) + jn) \bmod p \quad \text{for } 0 \leq j \leq p-1, \quad n \geq 1,$$

where $a(0) = \dots = a(p-1) = 0$, is called a *generalized Rudin–Shapiro sequence*.

Construction for a generalization of Rudin–Shapiro

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Remark

For the classical Rudin–Shapiro sequence we have the relations

$$r_{2n} = r_n \text{ and } r_{2n+1} = \begin{cases} r_n + 1, & \text{if } n \equiv 1 \bmod 2, \\ r_n, & \text{if } n \equiv 0 \bmod 2. \end{cases}$$

So, in the following definition with $p = 2$, we can see that the sequence is well the classical Rudin–Shapiro sequence.

Example (Tahay, 2020)

Let p be a prime number and $(a(n))_{n \geq 0}$ a generalized Rudin–Shapiro sequence. Let $F(t) = \sum_{n \geq 0} a(n)t^n$. We can show that for the polynomial

$$P(t, x) = \sum_{j=0}^{p-1} jt^j \sum_{i=0}^{p-1} it^{pi} + (t^{p^2} - 1)x + (1 - t^{p^2}) \sum_{j=0}^{p-1} t^j x^p,$$

we have $P(t, F(t)) = 0$.

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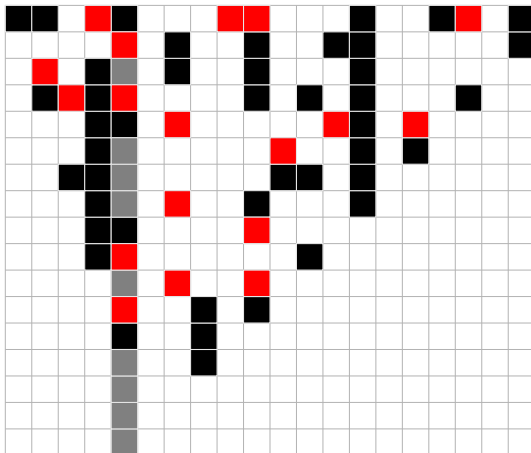
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we have $P(t, F(t)) = 0$.

Remark

- By the Christol's theorem, any generalized Rudin–Shapiro sequence is a p -automatic sequence.
- By the theorem of Rowland and Yassawi, all these sequences occurs in a column of a cellular automaton.

Generalized Rudin–Shapiro sequence for $p=3$



Proposition(Minsky-Papert criterion, 1966)

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be an increasing function and $\pi_f(x) = \#\{n : f(n) \leq x\}$. Minsky and Papert showed that if $\lim_{x \rightarrow \infty} \pi_f(x)/x = 0$ and $\lim_{n \rightarrow \infty} f(n+1)/f(n) = 1$, then the sequence $u = \mathbf{1}_{f(\mathbb{N})}$ is nonautomatic.

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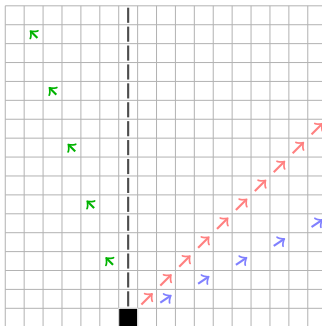
Let $\mathcal{P}$ be the set of primes numbers. The sequence $u = \mathbf{1}_{\mathcal{P}}$ is nonautomatic.

Example

If $P \in \mathbb{Q}[X]$ is a polynomial of degree $d \geq 2$ such that $P(\mathbb{N}) \subset \mathbb{N}$, and P is strictly increasing on $\mathbb{N}$, the sequence $\mathbf{1}_{P(\mathbb{N})}$ is nonautomatic.

Signals

In a spacetime diagram of a cellular automaton, it is possible to “transmit information” by using *signals*, that is to connect two cells (m, n) and $(m', n + t)$. We call *slope* of the signal the number $\frac{t}{m' - m}$.



■ vertical signal ■ slope 1 ■ slope -3 ■ slope $\frac{1}{2}$

Proposition (Marcovici, Stoll, Tahay, 2018)

Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be an ultimately strictly increasing function, and let g denote its summatory sequence defined by $g(n) = \sum_{k=0}^n f(k)$ for all $n \geq 0$. Suppose that $u = \mathbf{1}_{f(\mathbb{N})}$ belongs to $\mathcal{S}$. Then $v = \mathbf{1}_{g(\mathbb{N})}$ belongs to $\mathcal{S}$.

Construction of polynomial sequences

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Theorem (Marcovici, Stoll, Tahay, 2018)

Let $P(X) \in \mathbb{Q}[X]$ denote a polynomial of degree $d \geq 1$ with $P(\mathbb{N}) \subset \mathbb{N}$. Then the sequence $u = \mathbf{1}_{P(\mathbb{N})}$ belongs to $\mathcal{S}$ and can be realized by the hitting of one single signal in column 0.

Sketch of the proof (theorem)

If P has the form $P(X) = aX + b$ with $a \in \mathbb{N}$, $b \in \mathbb{Z}$, one can construct the sequence by the trajectory of a signal that hits a wall in column $\lfloor a/2 \rfloor$.

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Now, if P is a polynomial of degree d , with $d \geq 2$ and $P(\mathbb{N}) \subset \mathbb{N}$, then $Q(X) = P(X + 1) - P(X)$ is a polynomial of degree at most $d - 1$ with a positive leading coefficient.

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Moreover, we can observe that for all $n \geq 1$,

$$P(n) - P(0) = \sum_{k=0}^{n-1} (P(k+1) - P(k)) = \sum_{k=0}^{n-1} Q(k)$$

Sketch of the proof (theorem)

If P has the form $P(X) = aX + b$ with $a \in \mathbb{N}$, $b \in \mathbb{Z}$, one can construct the sequence by the trajectory of a signal that hits a wall in column $\lfloor a/2 \rfloor$.

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By induction, we know that $\mathbf{1}_{Q(\mathbb{N})}$ is constructible by a CA and with the previous proposition, $\mathbf{1}_{P(\mathbb{N})}$ belongs to $\mathcal{S}$.

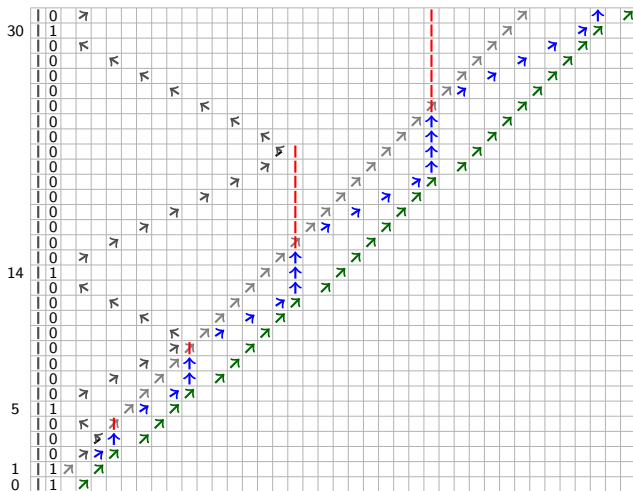
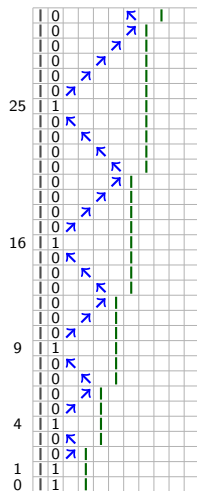
Example (cubes)

For all $k \in \mathbb{N}$ we have $(k+1)^3 - k^3 = 3k^2 + 3k + 1$.

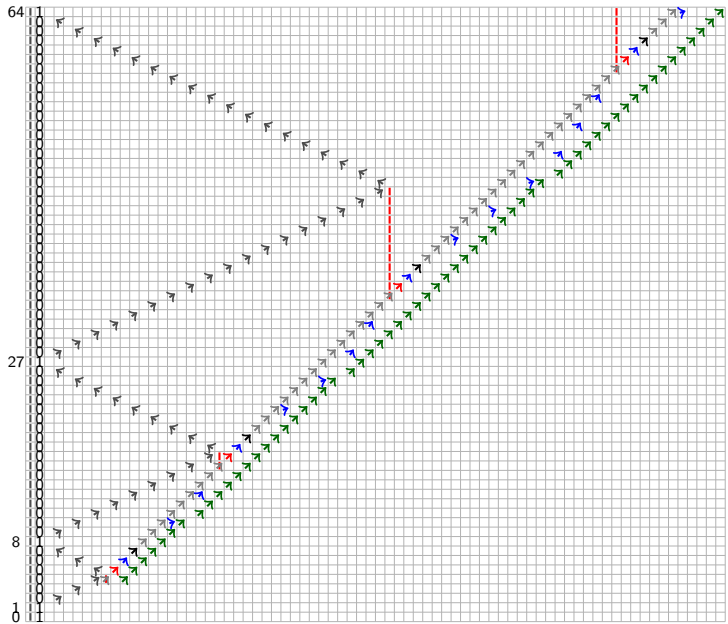
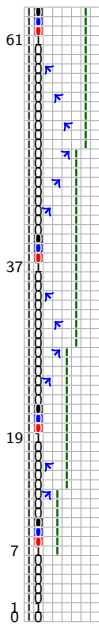
Then, for all $n \geq 1$, we have $\sum_{k=0}^{n-1} (3k^2 + 3k + 1) = n^3$.

From the construction of the characteristic function of the polynomial $3X^2 + 3X + 1$ of degree 2, we can construct the characteristic function of the cubes.

Construction of polynomial sequences



Squares (left) and the sum of the squares (right).



Difference of the cubes (left) and cubes (right).

Definition

Sturmian words are infinite words over a binary alphabet that have exactly $n + 1$ factors of length n for each non-negative n .

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Example

The Fibonacci word $\mathbf{f} = 010010100100101001010 \dots$ which is the unique fixed point of the substitution σ defined by $0 \mapsto 01$ and $1 \mapsto 0$ is a Sturmian word.

Continued fraction expansion

Let θ be a real number. A continued fraction expansion of θ is defined as $[c_0, c_1, c_2, \dots]$ whenever

$$\theta = c_0 + \frac{1}{c_1 + \frac{1}{c_2 + \ddots}}$$

with $c_0 \in \mathbb{Z}$ and $c_i \in \mathbb{Z}^+$ for every positive i .

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Example

The golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ is a quadratic irrational number. Its continued fraction expansion is $\varphi = [1] = [1, 1, 1, \dots]$

Directive sequences and Sturmian words

Definition

Let $\Delta = (d_n)_{n \geq 1}$ be an integer sequence with $d_1 \geq 0$ and $d_n > 0$ for every positive integer n . The *directive sequence* associated to Δ is the sequence of finite words $(w_n)_{n \geq -1}$ defined by

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Proposition

Every Sturmian word $\mathbf{w}$ can be obtained as a limit of a directive sequence $(w_n)_{n \geq 1}$ associated to $(d_1, d_2, d_3, \dots)$ such that $\alpha = [0, d_1 + 1, d_2, d_3, \dots]$ is an irrational number with $0 < \alpha < 1$. The number α is called the slope of the associated Sturmian word.

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Definition

We said that a real number α is quadratic if there is a polynomial P of degree 2 with integer coefficients such that $P(\alpha) = 0$.

Example

The Fibonacci word $\mathbf{f}$ can be obtained as the limit $\mathbf{f} = \lim_{n \rightarrow \infty} f_n$ where $(f_n)_{n \geq -1}$ is the directive sequence associated to $\Delta = (1^\omega) = (1, 1, 1, \dots)$.

$$f_{-1} = 1, f_0 = 0, \text{ and } f_n = f_{n-1}f_{n-2} \text{ for every } n \geq 1.$$

The continued fraction expansion of the corresponding irrational slope is $\frac{1}{\varphi^2} = [0, 2, \overline{1}]$

Theorem (Dolce, Tahay 2022)

A Sturmian word with quadratic slope can be represented as a column in the spacetime diagram of a one-dimensional cellular automaton.

Proposition

Let $(d_n)_{n \geq 1}$ be an eventually periodic integer sequence with $d_1 \in \mathbb{N}$ and $d_i \in \mathbb{Z}^+$ for every $i \geq 2$. Let $(S_n)_{n \geq 0}$ be the integer sequence defined by $S_n = d_n S_{n-1} + S_{n-2}$ for every $n \geq 0$, with $S_{-1}, S_0 > 0$. Then $\mathbf{1}_{\{S_n\}_{n \geq 0}}$ belongs to $\mathcal{S}$.

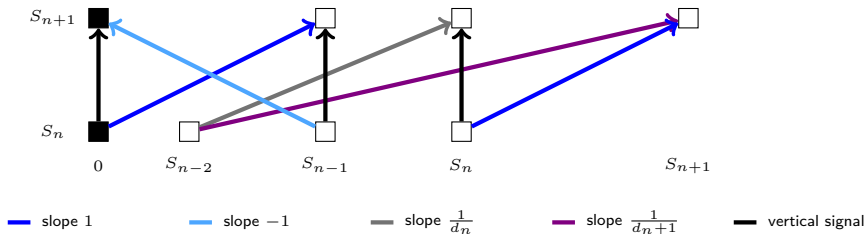
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Remark

Let $\mathbf{w}$ a Sturmian word obtained as the limit of the directive sequence $(w_n)_{n \geq 1}$ associated to $\Delta = (d_n)_{n \geq 1}$. Let $S_n = |w_n|$. So, we have the relation $S_n = d_n S_{n-1} + S_{n-2}$ for every $n \geq 0$.

Construction of the numbers when $d_{n+1} = 1$

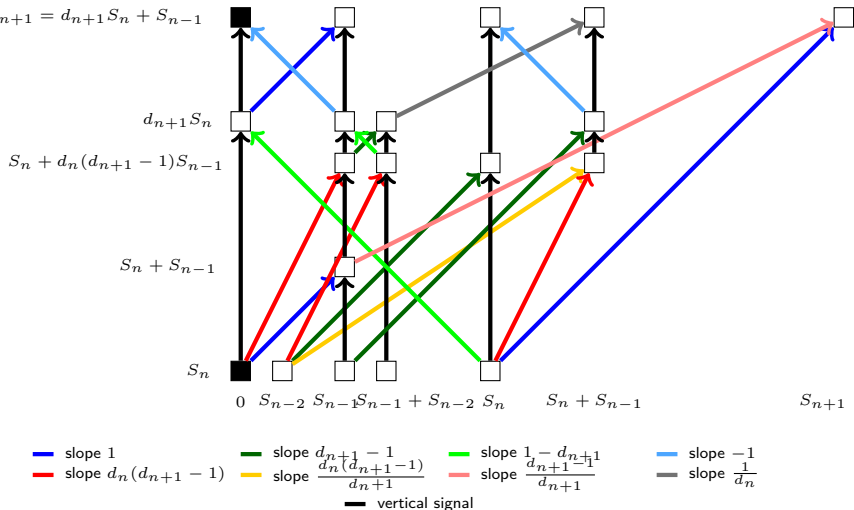


Construction of the numbers when $d_{n+1} \neq 1$

$$\begin{aligned} S_{n+1} &= d_{n+1}S_n + S_{n-1} \\ &= S_n + (d_{n+1} - 1)S_n + S_{n-1} \\ &= S_n + (d_{n+1} - 1)(d_n S_{n-1} + S_{n-2}) + S_{n-1} \\ &= S_n + d_n(d_{n+1} - 1)S_{n-1} + (d_{n+1} - 1)S_{n-2} + S_{n-1}. \end{aligned}$$

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$$S_{n+1} = d_{n+1} S_n + S_{n-1}$$



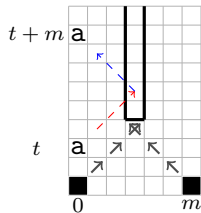
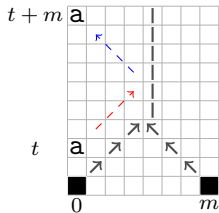
Proposition

Let $\Delta = (d_n)_{n \geq 1}$ be an eventually periodic integer sequence with $d_1 \geq 0$ and $d_n \geq 1$ for every $n > 1$; let $(w_n)_{n \geq -1}$ be the standard sequence associated to Δ defined by

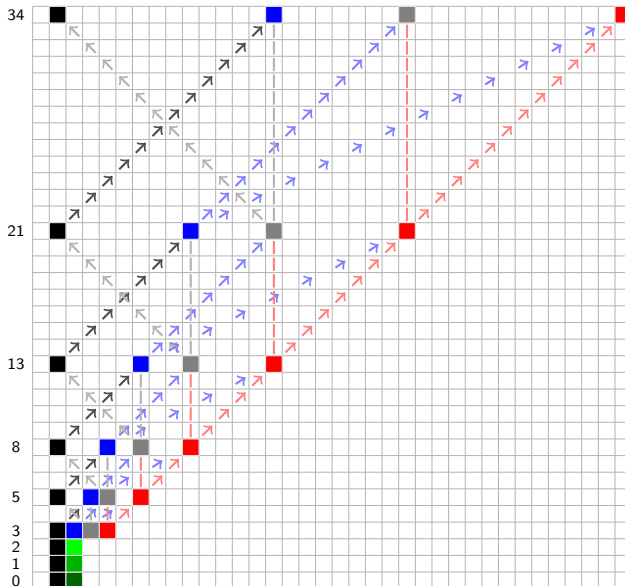
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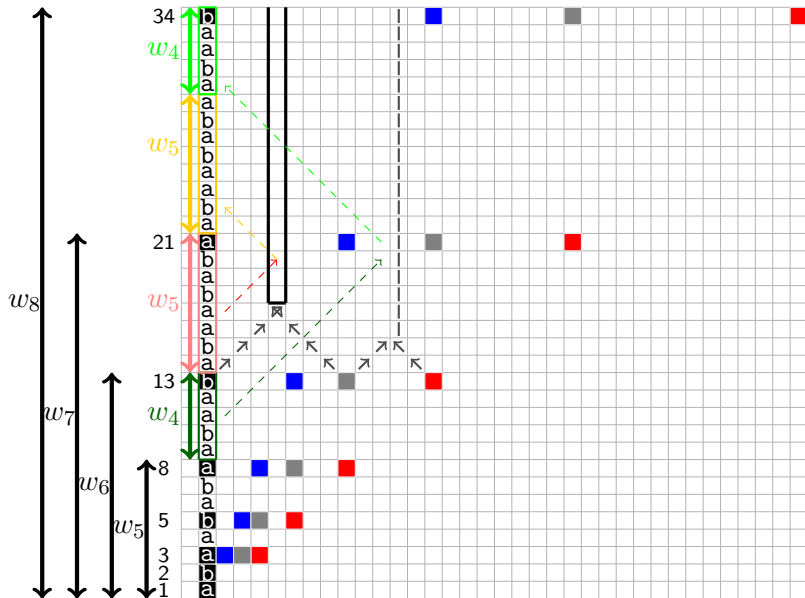
Recopying of letters



Construction of the Fibonacci numbers



Construction of the Fibonacci word



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- Which k -automatic sequences when k is not a prime power are constructible in a column of a CA?

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where $w^{(i)}$ is a word and $(\psi_i)_i \in \{G, D\}^{\mathbb{N}}$ with

$$G = \begin{cases} a \mapsto a \\ b \mapsto ab \end{cases} \quad \text{and} \quad D = \begin{cases} a \mapsto ba \\ b \mapsto b \end{cases}.$$

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Thank you for your attention!