

Some recent results on two “palindromicity” measures

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- Applications of Sturmian words in number theory, routing optimization, computer graphics and image processing, pattern recognition and more.
 - J.-P. Allouche & J. Shallit, *Automatic Sequences. Theory, Applications, Generalizations*, Cambridge University Press, Cambridge, 2003, Chapter 9.

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- music theory.

- T. Noll, Sturmian sequences and morphisms. A music-theoretical application, in A. Yves (ed.), *Mathématique et musique, Journée annuelle de la Société Mathématique de France*, 2008, pp. 79–102.
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- Clearly, different approaches can be imagined, depending on the interpretation of “more palindromic,” and various such measures have been defined in the literature.
- In this talk we present some of our results concerning two such measures.

Palindromic defect – definition

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Definition

For an infinite word w , its defect is defined by:

$$D(w) = \sup_{v \in \text{Fact}(w)} D(v).$$

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- However, infinite words of finite positive defect have been studied significantly less.
- One of the reasons for that is the fact that explicit constructions of such words (maybe with some additional constraints, such as aperiodicity, since periodic words are more-or-less straightforward to analyze) are somewhat deficient in the literature.
- For example, aperiodic words of finite positive defect, having the set of factors closed under reversal, had been deemed interesting from the point of view of some (then open) conjectures, but examples of such words were missing.

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- This suggests that uniformly recurrent words are in a way pretty “tame,” and that those that are not uniformly recurrent are less explored territory that deserves a closer look. One such word has been seen on the previous slide, and for a while it was the only known example (to the best of our knowledge).

Highly potential words – construction

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Let w be a finite word that is not a palindrome, and let c be a letter that does not occur in w . Define $w_0 = w$ and, for $i \in \mathbb{N}$,

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Finally, let

$$\text{hpw}(w) = \lim_{i \rightarrow \infty} w_i.$$

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Let $w, u, v \in \Sigma^*$, where $wuv \neq \varepsilon$ and u and v are palindromes, and let $A = (a_i)_{i \in \mathbb{N}}$ be a strictly increasing sequence of positive integers. We recursively define:

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- We believe that all this suggests that this new construction extends the class of highly potential words in a fairly noteworthy way.

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- In particular, this gives a characterization of generalized highly potential words of (non)zero defect.

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This upper bound is the best possible.

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- $r, s \in \{0, 1, \dots, n-1\}^*$, (r, s) is an MP-extension of $w \leftrightarrow rws$ is minimal-palindromic;
- (r, s) is an SMP-extension $\leftrightarrow |r| + |s|$ is the least possible;
- MP-ratio of the word w : $\frac{|rws|}{|w|}$, (r, s) – SMP-extension of w .

Definitions in the n -ary case

- w – n -ary word on $\{0, 1, \dots, n-1\}$;
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- $r, s \in \{0, 1, \dots, n-1\}^*$, (r, s) is an MP-extension of $w \leftrightarrow rws$ is minimal-palindromic; – Does it always exist???
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Well-definedness for $n \geq 4$

Theorem

Each n -ary word has an MP-extension.

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Let $M = 2^{\lceil \frac{n}{2} \rceil} + 2^{\lfloor \frac{n}{2} \rfloor - 1} - 3$; exceptionally, if $n = 4$ or $n = 5$, we define $M = 4$, respectively $M = 8$ instead.

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$$r = 0^{l_0} 1^{l_1} \dots (n-2)^{l_{n-2}}, \quad s = 1^{r_1} 2^{r_2} \dots (n-1)^{r_{n-1}},$$

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where $l_0 = M|w| - |w|_0$ and

$$l_i = \begin{cases} (M - 2^i + 1)|w|, & \text{for } i = 1, \dots, \lceil \frac{n}{2} \rceil - 1; \\ (2^{n-1-i} - 1)|w| - |w|_i, & \text{for } i = \lceil \frac{n}{2} \rceil, \dots, n-2, \end{cases}$$

while $r_{n-1} = M|w| - |w|_{n-1}$ and

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Then (r, s) is an MP-extension of w . ■

Well-definedness for $n \geq 4$

Example

For the 4-ary word 01231 our construction gives the MP-extension

Well-definedness for $n \geq 4$

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$$0^{19} \ 1^{15} \ 2^4 \ 01231 \ 1^3 \ 2^{15} \ 3^{19}.$$

An upper bound on the MP-ratio in the n -ary case, $n \geq 4$

Theorem

Let w be an n -ary word for $n \geq 4$, and let

$$M = \begin{cases} 4, & \text{if } n = 4; \\ 8, & \text{if } n = 5; \\ 2^{\lceil \frac{n}{2} \rceil} + 2^{\lfloor \frac{n}{2} \rfloor} - 3, & \text{if } n \geq 6. \end{cases}$$

Then the MP-ratio of w is not greater than nM .

On optimality

Theorem

The optimal upper bound on the MP-ratio is not less than $2n$.

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Proof (sketch).

$\approx 0^{2 w }$	$\approx 1^{2 w }$	$\approx 2^{2 w }$	$\dots$	$\approx (n-1)^{2 w }$
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On optimality

Now we know that the optimal upper bound on the MP-ratio for $n \geq 4$ is somewhere between

$$2n \quad \text{and} \quad \sim 2^{\frac{n}{2}} n,$$

and it remains an open problem to narrow (or even better, close) this gap.

Ternary case – the main theorem

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$$f(w) = 0^{2|w|-|w|_0} 2^{2|w|-|w|_2-g'(w,0,2)} w 2^{g'(w,0,2)} 1^{2|w|-|w|_1}$$

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$g'(w, a, b) = \max(\{2|w[i, |w|]|_a - |w[i, |w|]|_b : i = 1, 2, \dots, j(a, w)\} \cup \{0\})$, where $j(a, w)$ denotes the position of the last occurrence of a in w , and $j(a, w) = 0$ if a does not occur in w .

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- We prove: Assuming (w.l.o.g.) $|w|_0 \leq |w|_1 \leq |w|_2$, at least one of $f(w)$ and $f'(w)$ is minimal-palindromic.

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- We prove: Assuming (w.l.o.g.) $|w|_0 \leq |w|_1 \leq |w|_2$, at least one of $f(w)$ and $f'(w)$ is minimal-palindromic.
- In other words: At least one of them does not have subpalindromes whose length exceeds $2|w|$.

Subpalindromes of the form $0p0$ and $1p1$

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The length of an arbitrary subpalindrome of the form $0p0$ in each of the words $f(w)$ and $f'(w)$ is less than or equal to $2|w|$.

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Subpalindromes of the form $2p2$ – pt. 1/9

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Theorem

Let $u \in \{1, 2\}^*$, $t, v \in 2^*$ and let p and q be subpalindromes of tu and uv , respectively. If $|p| + |q| > 2|u|$,

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Subpalindromes of the form $2p2$ – pt. 2/9

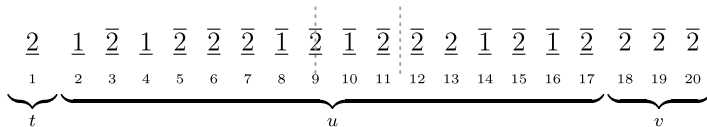
Proof of the previous theorem (sketch). We define $\sigma_P : P \rightarrow P$ by $\sigma_P : P_s \mapsto P_{|P|-s+1}$ and $\sigma_Q : Q \rightarrow Q$ by $\sigma_Q : Q_s \mapsto Q_{|Q|-s+1}$. Note that σ_P and σ_Q are bijections and their squares are identical mappings. For $1 \leq n \leq |t|$ let $\sigma_0(n) = n$ and

$$\sigma_{i+1}(n) = \begin{cases} \sigma_P(\sigma_i(n)), & \text{for } 2 \mid i \text{ and } \sigma_i(n) \in P; \\ \sigma_Q(\sigma_i(n)), & \text{for } 2 \nmid i \text{ and } \sigma_i(n) \in Q; \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

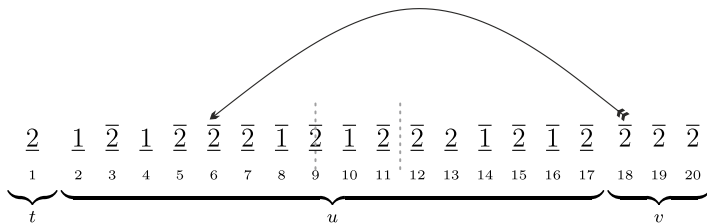
In a similar manner, for $|tu| + 1 \leq n \leq |tuv|$ let $\sigma_0(n) = n$ and

$$\sigma_{i+1}(n) = \begin{cases} \sigma_Q(\sigma_i(n)), & \text{for } 2 \mid i \text{ and } \sigma_i(n) \in Q; \\ \sigma_P(\sigma_i(n)), & \text{for } 2 \nmid i \text{ and } \sigma_i(n) \in P; \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

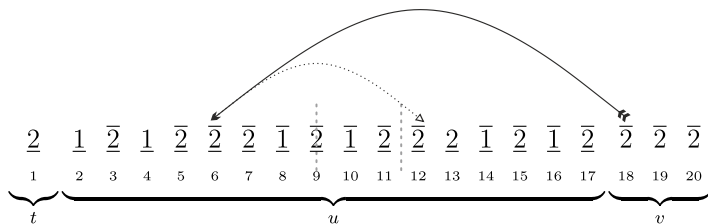
Subpalindromes of the form $2p2$ – pt. 3/9



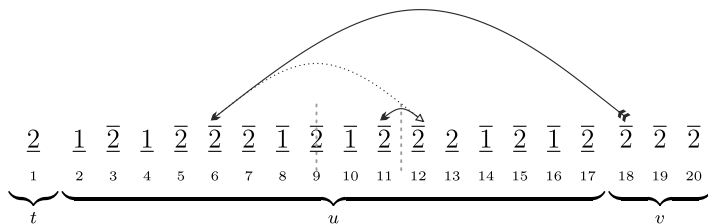
Subpalindromes of the form $2p2$ – pt. 3/9



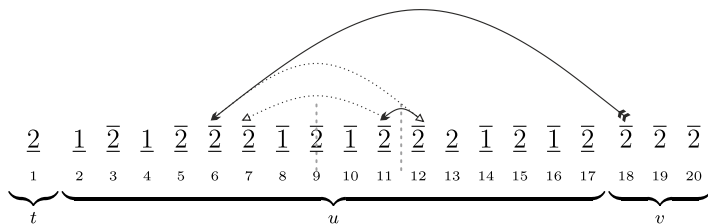
Subpalindromes of the form $2p2$ – pt. 3/9



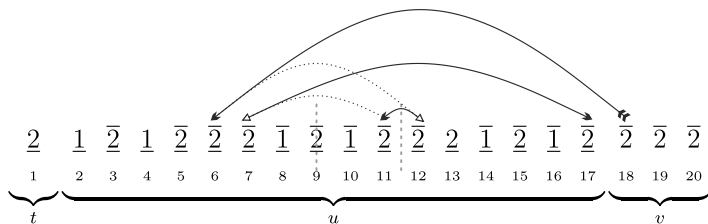
Subpalindromes of the form $2p2$ – pt. 3/9



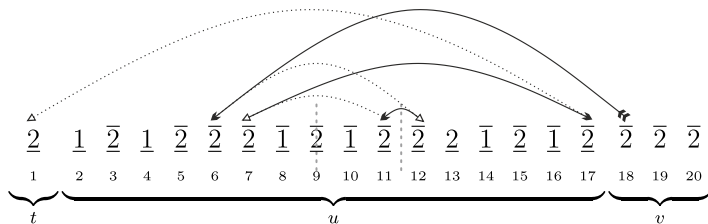
Subpalindromes of the form $2p2$ – pt. 3/9



Subpalindromes of the form $2p2$ – pt. 3/9



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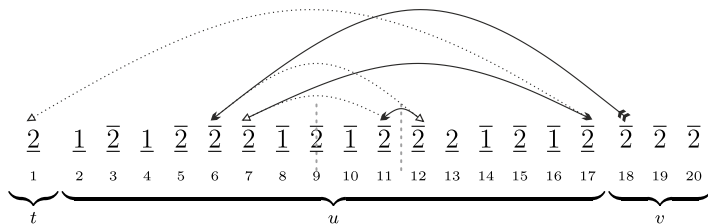


Subpalindromes of the form $2p2$ – pt. 4/9

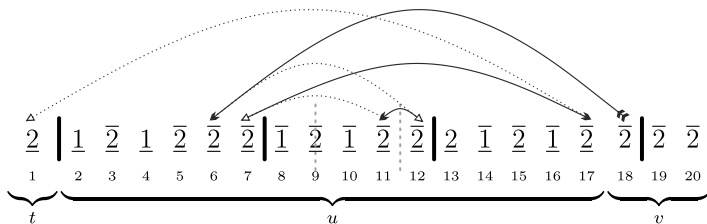
Lemma

- a) For every $n, m \in Q$ (respectively, $n, m \in P$), if $n < m$, then $\sigma_Q(n) > \sigma_Q(m)$ (respectively, $\sigma_P(n) > \sigma_P(m)$).
- b) We have $\sigma_0(n) > \sigma_2(n) > \sigma_4(n) > \dots$ and $\sigma_1(n) < \sigma_3(n) < \sigma_5(n) < \dots$ for $n \geq |tu| + 1$, and $\sigma_0(n) < \sigma_2(n) < \sigma_4(n) < \dots$ and $\sigma_1(n) > \sigma_3(n) > \sigma_5(n) > \dots$ for $n \leq |t|$.
- c) For each n such that $n \leq |t|$ or $n \geq |tu| + 1$, there exists $z \in \mathbb{N}$ such that $\sigma_z(n)$ is the last defined term in the sequence $\sigma_0(n), \sigma_1(n), \sigma_2(n), \dots$.

Subpalindromes of the form $2p2$ – pt. 5/9



Subpalindromes of the form $2p2$ – pt. 5/9



Subpalindromes of the form $2p2$ – pt. 6/9

Lemma

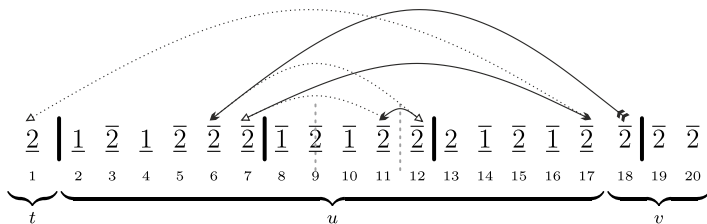
a) For each $n \in Q$ such that $\sigma_P(\sigma_Q(n))$ is defined (that is, $\sigma_Q(n) \in P$), we have:

$$n - \sigma_P(\sigma_Q(n)) \leq 2(|t| + |v|) - 1 - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q| \\ - |[1, \sigma_Q(n)]_{\mathbb{N}} \setminus P| - |[1, \sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P|.$$

b) For each $n \in Q$ such that $\sigma_P(\sigma_Q(n))$ is undefined (that is $\sigma_Q(n) \notin P$), we have:

$$n \leq 2(|t| + |v|) + |P| - \sigma_Q(n) - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q|.$$

Subpalindromes of the form $2p2$ – pt. 7/9



Subpalindromes of the form $2p2$ – pt. 8/9

Lemma

Let $i, i \geq 0$, be such that $\sigma_{2i+2}(n_0)$ is defined.

a) For each $m, m \geq |tu| + 1$, we have one of the following:

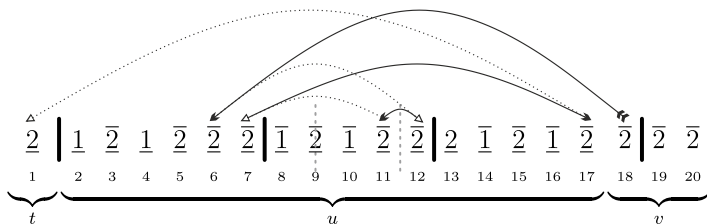
- there exists j such that $\sigma_{2i+2}(n_0) < \sigma_j(m) \leq \sigma_{2i}(n_0)$;
- $2 \mid \text{end}(m)$ and $\sigma_{\text{end}(m)}(m) > \max\{\sigma_{2i}(n_0), \sigma_{2i+1}(n_0)\}$;
- $2 \nmid \text{end}(m)$ and $\sigma_{\text{end}(m)}(m) < \min\{\sigma_{2i+1}(n_0), \sigma_{2i+2}(n_0)\}$.

Also, for each $m, m \leq |t|$, we have one of the following:

- there exists j such that $\sigma_{2i+2}(n_0) < \sigma_j(m) \leq \sigma_{2i}(n_0)$;
- $2 \nmid \text{end}(m)$ and $\sigma_{\text{end}(m)}(m) > \max\{\sigma_{2i}(n_0), \sigma_{2i+1}(n_0)\}$;
- $2 \mid \text{end}(m)$ and $\sigma_{\text{end}(m)}(m) < \min\{\sigma_{2i+1}(n_0), \sigma_{2i+2}(n_0)\}$.

b) To each $m, m \leq |t|$ or $m \geq |tu| + 1$, for which there exists j described in a) we can assign one such j in such a way that all the corresponding $\sigma_j(m)$ are different.

Subpalindromes of the form $2p2$ – pt. 9/9



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