

Balancedness constants of words generated by billiards in the hypercube

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One World Combinatorics on Words Seminar
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I. Settings and motivations

II. Momenta with rationally independent coordinates

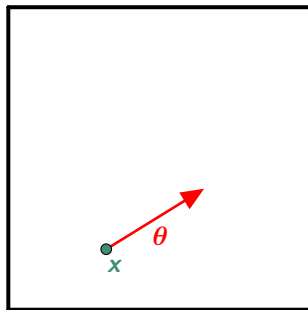
III. Momenta with rationally dependent coordinates

I. Settings and motivations

Hypercubic billiard

Let $d \geq 1$.

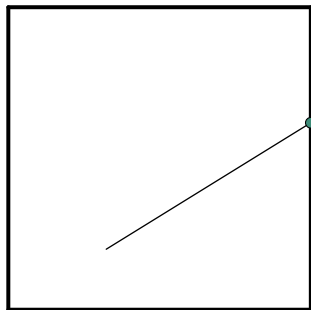
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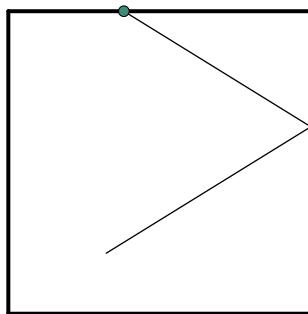
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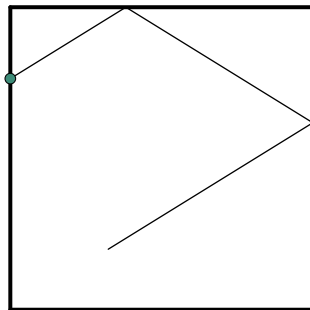
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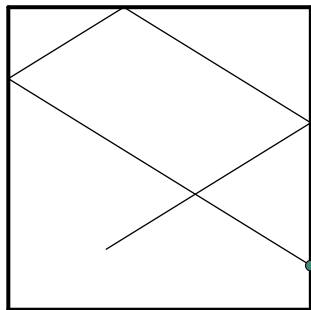
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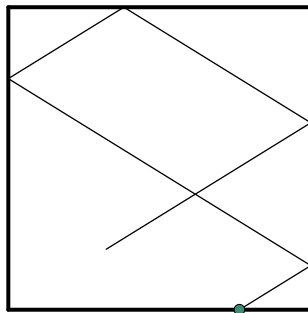
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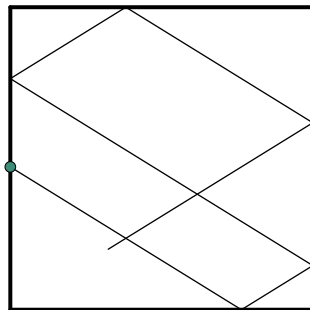
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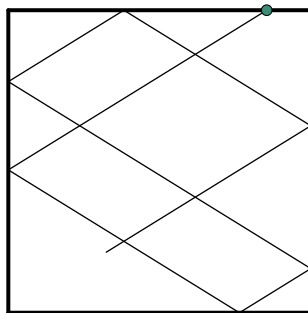
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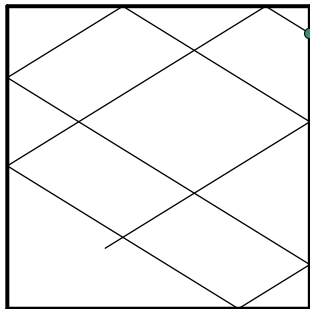
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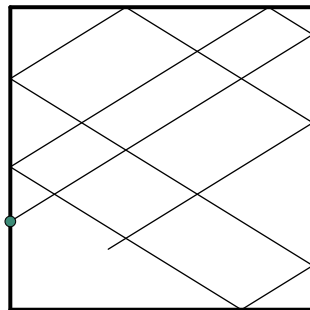
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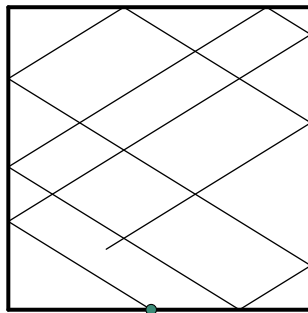
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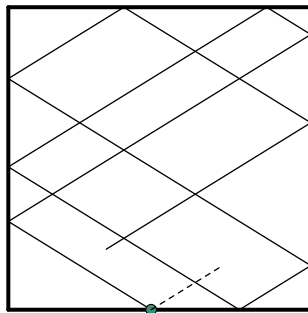
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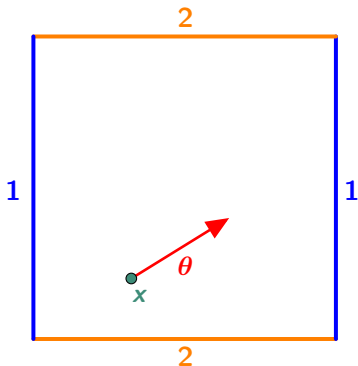
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Hypercubic billiard words



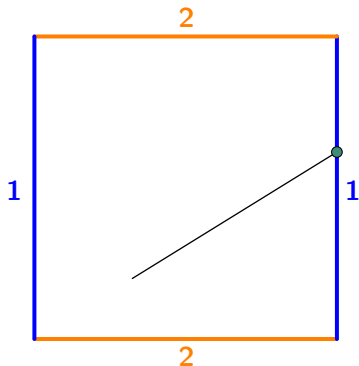
$$w(x, \theta) =$$

Def: The **hypercubic billiard word** with initial *position* x and initial *momentum* θ is:

$$w(x, \theta) \in \{1, \dots, d\}^{\mathbb{N}}$$

the **sequence of the faces successively hit by the ball** for **positive** time.

Hypercubic billiard words



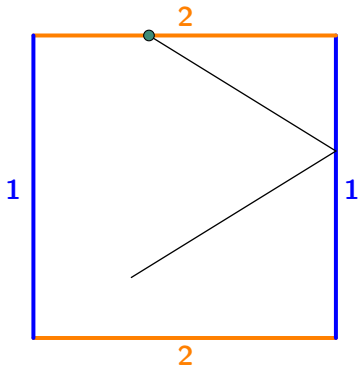
$$w(x, \theta) = 1$$

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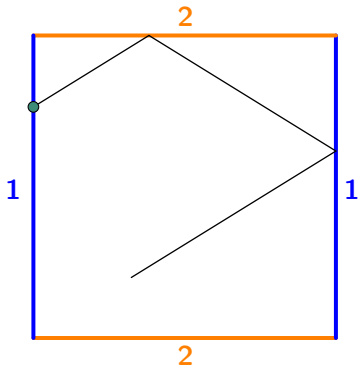
$$w(x, \theta) = 12$$

Def: The **hypercubic billiard word** with initial *position* x and initial *momentum* θ is:

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Hypercubic billiard words



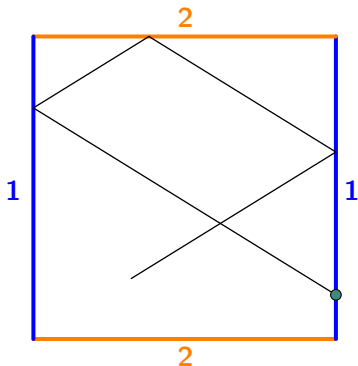
$$w(x, \theta) = 121$$

Def: The **hypercubic billiard word** with initial *position* x and initial *momentum* θ is:

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Hypercubic billiard words



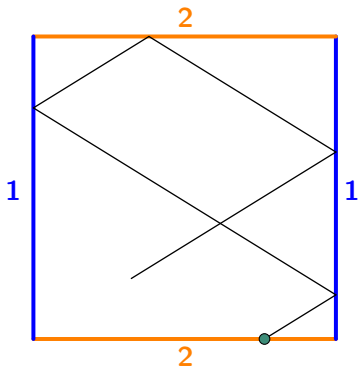
$$w(x, \theta) = 1211$$

Def: The **hypercubic billiard word** with initial *position* x and initial *momentum* θ is:

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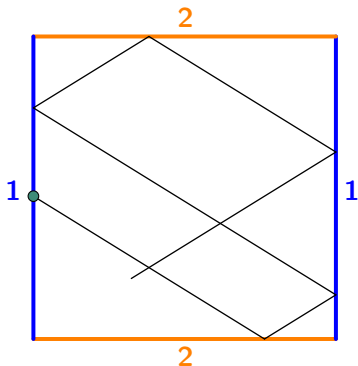
$$w(x, \theta) = 12112$$

Def: The **hypercubic billiard word** with initial *position* x and initial *momentum* θ is:

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Hypercubic billiard words



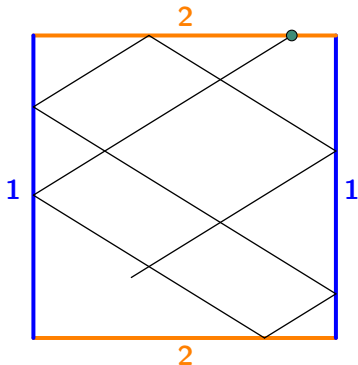
$$w(x, \theta) = 121121$$

Def: The **hypercubic billiard word** with initial *position* x and initial *momentum* θ is:

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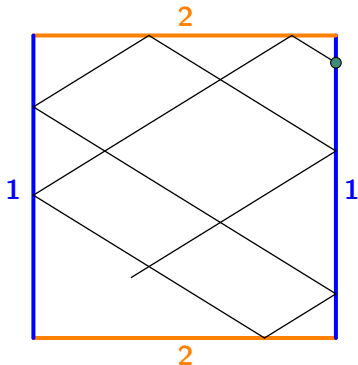
$$w(x, \theta) = 1211212$$

Def: The **hypercubic billiard word** with initial *position* x and initial *momentum* θ is:

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Hypercubic billiard words



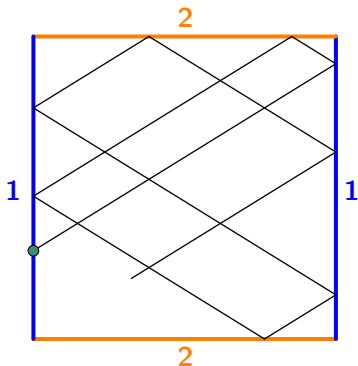
$$w(x, \theta) = 12112121$$

Def: The **hypercubic billiard word** with initial *position* x and initial *momentum* θ is:

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Hypercubic billiard words



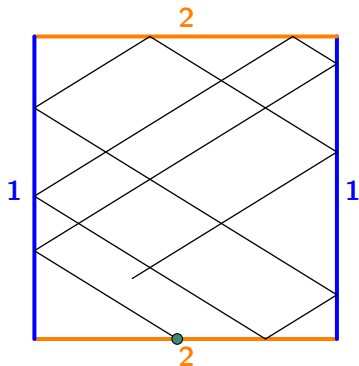
$$w(x, \theta) = 121121211$$

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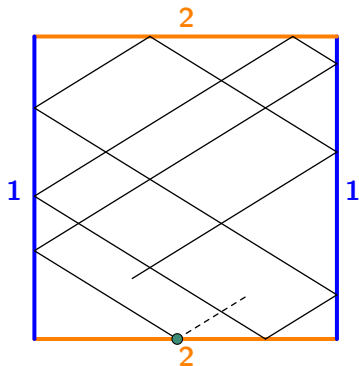
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Hypercubic billiard words



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An important remark

Classical Result: The trajectory of the ball is **dense** in the hypercube if, and only if, the momentum θ has **R.I.** coordinates.

- ***R.I.** = **rationally independent** = the $\mathbb{Q}$ -linear space generated by the coordinates of θ has dimension d
- ***R.D.** = **rationally dependent** = the $\mathbb{Q}$ -linear space generated by the coordinates of θ has dimension $\delta < d$

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Consequence: If θ has **R.I.** coordinates, then, for every initial positions x, y

$$\mathcal{L}(w(x, \theta)) = \mathcal{L}(w(y, \theta))$$

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Consequence: If θ has **R.I.** coordinates, then, for every initial positions x, y

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Rk: This result is false when θ has **R.D.** coordinates.

Ex: Take $\theta = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $x = (\frac{1}{2}, \frac{1}{8}, \frac{1}{16})$ and $y = (\frac{1}{2}, \frac{1}{3}, \frac{1}{16})$. Then

$$w(x, \theta) = 1123 \cdot 1123 \cdot 1123 \dots$$

$$w(y, \theta) = 1213 \cdot 1213 \cdot 1213 \dots$$

A second remark

We lose no generality by choosing momenta in

$$\Delta_d := \left\{ (\theta_1, \dots, \theta_d) \in (\mathbb{R}_{>0})^d \mid \sum_{i=1}^d \theta_i = 1 \right\}$$

↪ In this case, $\theta_1, \theta_2, \dots, \theta_d$ are the **letter frequencies** of the billiard word.

Square billiard and Sturmian words

Fact [Morse, Hedlund 40]

$$\{\text{Sturmian words}\} = \{\text{Square billiard words with } \theta \text{ R.I.}^*\}$$

* θ R.I. = the coordinates of the momentum θ are **Rationally Independent**.

Rk: Hypercubic billiard words, with R.I. momenta, are a **dynamical generalization of Sturmian words**.

Imbalance

Notation: $|u|_a$ = number of occurrences of the letter a in the finite word u .

Ex: $|abaca|_a = 3$

Def: A word w is **C-balanced** if, for every equally long factors $u, v \in \mathcal{L}(w)$, and every letter a ,

$$||u|_a - |v|_a| \leq C.$$

The **imbalance** of a word w is

$$\text{imb}(w) := \inf \{ C \mid w \text{ is } C\text{-balanced} \} \in \mathbb{N} \cup \{+\infty\}$$

Ex:

- $w_1 = 1111111111\dots$ $\text{imb}(w_1) = 0$
- $w_2 = 0 \cdot 1 \cdot 00 \cdot 01 \cdot 10 \cdot 11 \cdot 000 \cdot 001\dots$ $\text{imb}(w_2) = +\infty$

Initial motivation

In dim 2, the imbalance characterizes billiard words:

Theorem [Morse, Hedlund 40]

- i. $\left\{ \begin{array}{l} \text{Square billiard words with} \\ \text{R.I. letter frequencies} \end{array} \right\} = \left\{ \begin{array}{l} \text{Non eventually-periodic} \\ \text{binary words with imbalance 1} \end{array} \right\}$
- ii. $\left\{ \begin{array}{l} \text{Square billiard words with} \\ \text{R.D. letter frequencies} \end{array} \right\} = \left\{ \begin{array}{l} \text{Purely-periodic} \\ \text{binary words with imbalance 1} \end{array} \right\}$

Question

What can we say in higher dimension ?

II. Momenta with **rationally independent** coordinates

Known results

Theorem [Vuillon 03]

Let $\theta \in \Delta_d$ with R.I. entries.

Then, for every initial position x , $w(x, \theta)$ is $(d - 1)$ -balanced.

Theorem [Vuillon 03]

There exists $\theta \in \Delta_d$ with R.I. entries s.t., for every initial position x ,

$$\text{imb}(w(x, \theta)) = d - 1.$$

Initial questions

Question

- 1) When $\theta \in \Delta_d$ has R.I. entries, do we always have $\text{imb}(w(x, \theta)) = d - 1$?
- 2) If no, can we determine for which integers k there exists $\theta \in \Delta_d$ with R.I. entries and s.t. $\text{imb}(w(x, \theta)) = k$?
- 3) If yes, can we characterize the momenta $\theta \in \Delta_d$ with R.I. entries s.t. $\text{imb}(w(x, \theta)) = k$?
- 4) Or, at least, can we characterize the momenta $\theta \in \Delta_d$ with R.I. entries s.t. $\text{imb}(w(x, \theta)) = d - 1$?

First results

Theorem 1 [Andrieu, V. 22+]

Let $\theta \in \Delta_d$ with R.I. entries.

- i. If $d \in \{1, 2, 3, 4\}$, then, for every initial position x ,

$$\text{imb}(w(x, \theta)) = d - 1.$$

- ii. If $d \geq 5$, then, for every initial position x ,

$$\text{imb}(w(x, \theta)) \in \{3, 4, \dots, d - 1\}.$$

Theorem 2 [Andrieu, V. 22+]

Let $d \geq 5$. For every $k \in \{3, 4, \dots, d - 1\}$, there exists $\theta \in \Delta_d$ with R.I. entries s.t., for every initial position x , $\text{imb}(w(x, \theta)) = k$.

Initial questions

Question

- 1) When $\theta \in \Delta_d$ has R.I. entries, do we always have $\text{imb}(w(x, \theta)) = d - 1$? **No!**
- 2) If no, can we determine for which integers k there exists $\theta \in \Delta_d$ with R.I. entries and s.t. $\text{imb}(w(x, \theta)) = k$? **Yes!**
- 3) If yes, can we characterize the momenta $\theta \in \Delta_d$ with R.I. entries s.t. $\text{imb}(w(x, \theta)) = k$?
- 4) Or, at least, can we characterize the momenta $\theta \in \Delta_d$ with R.I. entries s.t. $\text{imb}(w(x, \theta)) = d - 1$?

A master theorem for R.I. momenta

Theorem 3 [Andrieu, V. 22+]

Let $d \geq 3$ and $\theta \in \Delta_d$ with R.I. entries. Then, for every initial position x ,

$$\text{imb}(w(x, \theta)) = 2 + \max_{i \text{ letter}} |\text{pref}_{d-3}(w(\mathbf{0}, \theta))|_i$$

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Consequences:

- $\text{imb}(w(x, \theta)) \leq 2 + d - 3 = d - 1$

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- If $d \geq 4$, then $\text{imb}(w(x, \theta)) \geq 2 + 1 = 3$.

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- If $d = 4$, then $\text{imb}(w(x, \theta)) = 2 + 1 = 3$.
- If $d \geq 4$, then $\text{imb}(w(x, \theta)) \geq 2 + 1 = 3$.
- Consider $\theta \in \Delta_d$ with R.I. entries s.t. $\theta_1 \gg \theta_i$. Then $\text{pref}_{d-3}(w(0, \theta)) = 1^{d-3}$ and $\text{imb}(w(x, \theta)) = d - 1$.

A master theorem for R.I. momenta

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Consequences:

- $\text{imb}(w(x, \theta)) \leq 2 + d - 3 = d - 1$
- If $d = 3$, then $\text{imb}(w(x, \theta)) = 2 + 0 = 2$.
- If $d = 4$, then $\text{imb}(w(x, \theta)) = 2 + 1 = 3$.
- If $d \geq 4$, then $\text{imb}(w(x, \theta)) \geq 2 + 1 = 3$.
- Consider $\theta \in \Delta_d$ with R.I. entries s.t. $\theta_1 \gg \theta_i$. Then $\text{pref}_{d-3}(w(0, \theta)) = 1^{d-3}$ and $\text{imb}(w(x, \theta)) = d - 1$.
- Consider $\theta \in \Delta_d$ with R.I. entries s.t. $\theta_1 \simeq \theta_2 \simeq \dots \simeq \theta_d$. Then $\text{pref}_d(w(0, \theta)) = 12 \dots d$ and $\text{imb}(w(x, \theta)) = 2 + 1 = 3$.

A master theorem for R.I. momenta

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- If $d = 3$, then $\text{imb}(w(x, \theta)) = 2 + 0 = 2$.
- If $d = 4$, then $\text{imb}(w(x, \theta)) = 2 + 1 = 3$.
- If $d \geq 4$, then $\text{imb}(w(x, \theta)) \geq 2 + 1 = 3$.
- Consider $\theta \in \Delta_d$ with R.I. entries s.t. $\theta_1 \gg \theta_i$. Then $\text{pref}_{d-3}(w(0, \theta)) = 1^{d-3}$ and $\text{imb}(w(x, \theta)) = d - 1$.
- Consider $\theta \in \Delta_d$ with R.I. entries s.t. $\theta_1 \simeq \theta_2 \simeq \dots \simeq \theta_d$. Then $\text{pref}_d(w(0, \theta)) = 12 \dots d$ and $\text{imb}(w(x, \theta)) = 2 + 1 = 3$.

Vuillon's theorems and Theorem 1 and 2 are proven from the master theorem!

Initial questions are answered!

Question

- 1) When $\theta \in \Delta_d$ has R.I. entries, do we always have $\text{imb}(w(x, \theta)) = d - 1$? **No!**
- 2) If no, can we determine for which integers k there exists $\theta \in \Delta_d$ with R.I. entries and s.t. $\text{imb}(w(x, \theta)) = k$? **Yes!**
- 3) If yes, can we characterize the momenta $\theta \in \Delta_d$ with R.I. entries s.t. $\text{imb}(w(x, \theta)) = k$? **Yes!**
- 4) Or, at least, can we characterize the momenta $\theta \in \Delta_d$ with R.I. entries s.t. $\text{imb}(w(x, \theta)) = d - 1$? **Yes!**

Partition of the simplex Δ_d

Proposition 4 [Andrieu, V. 22+]

Let $d \geq 5$. For every $k \in \{3, \dots, d-1\}$, let

$$\mathcal{R}_k := \overline{\{\theta \in \Delta_d \text{ R.I.} \mid \text{imb}(w(\cdot, \theta)) = k\}}^{\Delta_d}.$$

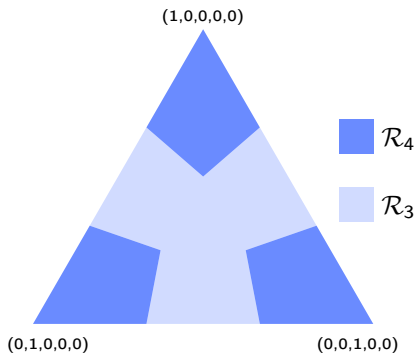
Then, each $\mathcal{R}_k$ is a **union of polytopes**, and the $\mathcal{R}_k$ **"partition"** Δ_d .

Ex: $d = 5$ and $k = d - 1 = 4$

$$\begin{aligned} \mathcal{R}_4 &= \overline{\{\theta \in \Delta_5 \text{ R.I.} \mid \text{pref}_2(w(0, \theta)) = ii \text{ } i \text{ letter}\}}^{\Delta_5} \\ &= \bigcup_{i=1}^5 \{\theta \in \Delta_d \mid \theta_i \geq 2\theta_j, j \in \{1, 2, 3, 4, 5\} \setminus \{i\}\} \end{aligned}$$

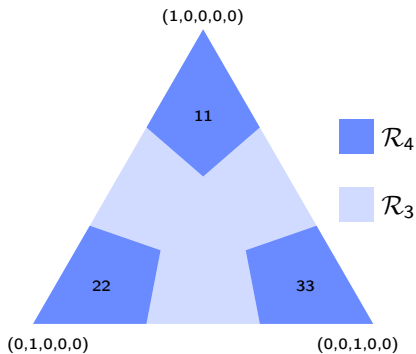
Reminder:

$$\text{imb}(w(x, \theta)) = 2 + \max_{i \text{ letter}} |\text{pref}_{d-3}(w(0, \theta))|_i$$

Example: cut views of Δ_d for $d = 5$ 

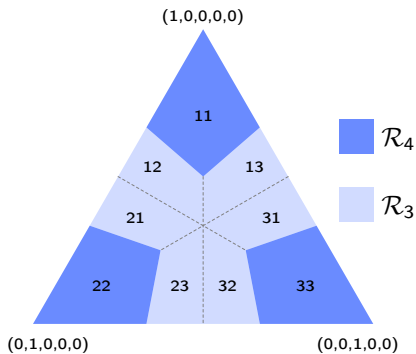
$$\mathcal{R}_4 = \bigcup_{i=1}^5 \bigcap_{\substack{j=1 \\ j \neq i}}^5 \{ \theta \in \Delta_d \mid \theta_i \geq 2\theta_j \}$$

2d-cut of Δ_5 ($\theta_4 \simeq \theta_5 \simeq 0$)

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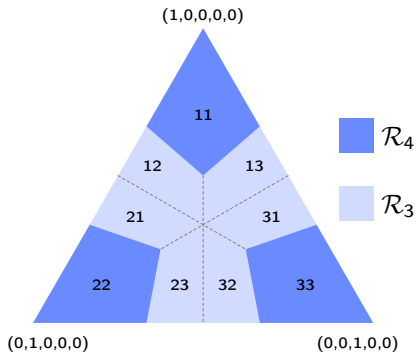
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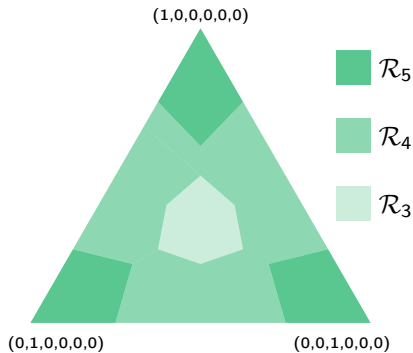
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2d-cut of Δ_5 ($\theta_4 \simeq \theta_5 \simeq 0$)

Example: cut views of Δ_d for $d = 5, 6$

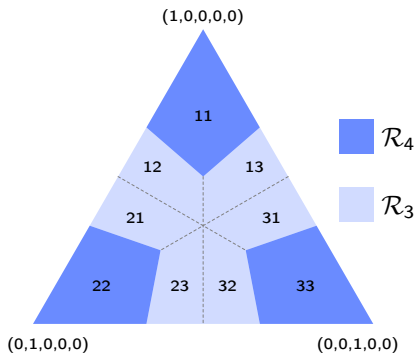


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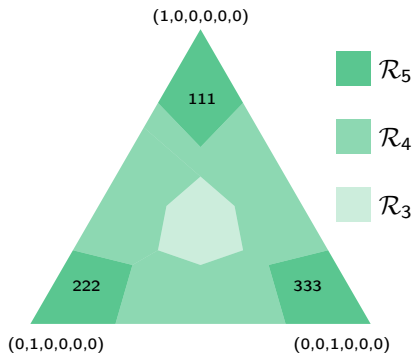


2d-cut of Δ_6 ($\theta_4 \simeq \theta_5 \simeq \theta_6 \simeq 0$)

Example: cut views of Δ_d for $d = 5, 6$

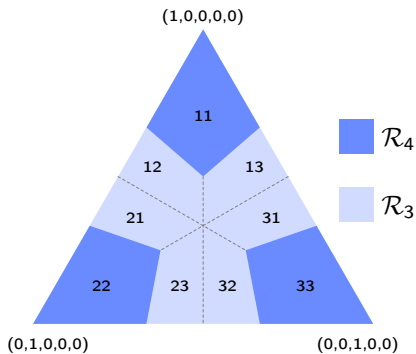


2d-cut of Δ_5 ($\theta_4 \simeq \theta_5 \simeq 0$)

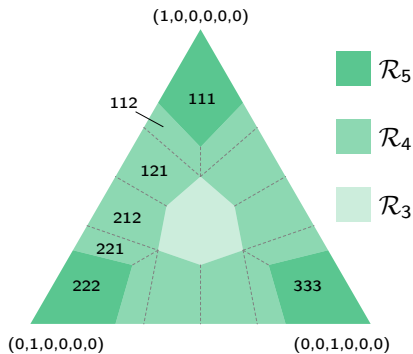


2d-cut of Δ_6 ($\theta_4 \simeq \theta_5 \simeq \theta_6 \simeq 0$)

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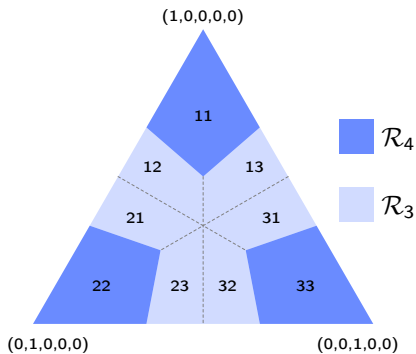


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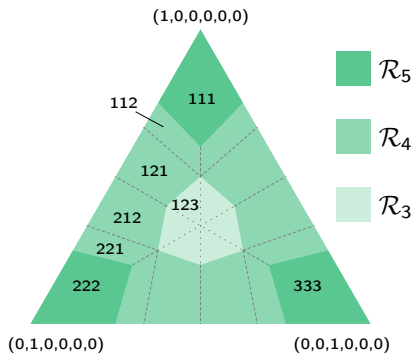


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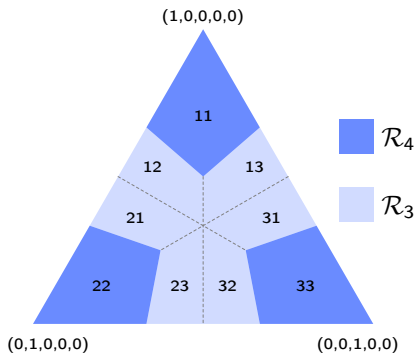


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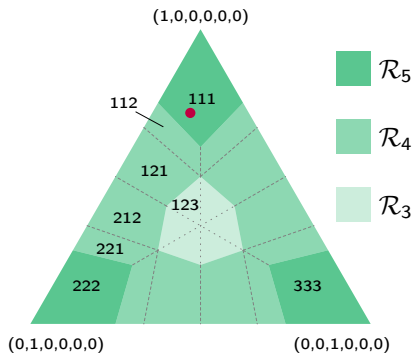


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2d-cut of Δ_6 ($\theta_4 \simeq \theta_5 \simeq \theta_6 \simeq 0$)

III. Momenta with rationally dependent coordinates

New questions

Questions

What happens when the coordinates of the momentum θ are **rationally dependent**?

- 1) How does the imbalance depend on $\mathcal{R}_k$?
- 2) Does the imbalance depend on the $\mathbb{Q}$ -dimension of θ ?

First results for R.D. momenta

Theorem 5 [Andrieu, V. 24+]

- i. For every momentum $\theta \in \mathcal{R}_k$ and every initial position x , we have

$$\text{imb}(\mathbf{w}(x, \theta)) \leq k.$$

- ii. Moreover, around each $\theta \in \mathcal{R}_k$ with R.I. entries, there exists a neighborhood $\mathcal{N}_\theta$ s.t.

$$\forall \theta' \in \mathcal{N}_\theta \text{ and } \forall x \in [0, 1]^d, \quad \text{imb}(\mathbf{w}(x, \theta')) = k.$$

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$$\forall \theta' \in \mathcal{N}_\theta \text{ and } \forall x \in [0, 1]^d, \quad \text{imb}(w(x, \theta')) = k.$$

BUT: not all $\theta \in \mathcal{R}_k$ generate words with imbalance k .

WORST: the imbalance sometimes depends on the starting position $x \in [0, 1]^d$.

Ex: Take again $\theta = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $x = (\frac{1}{2}, \frac{1}{8}, \frac{1}{16})$ and $y = (\frac{1}{2}, \frac{1}{3}, \frac{1}{16})$. Then

$$\begin{aligned} w(x, \theta) &= 1123 \cdot 1123 \cdot 1123 \dots & \text{imb}(w(x, \theta)) &= 2 \\ w(y, \theta) &= 1213 \cdot 1213 \cdot 1213 \dots & \text{imb}(w(y, \theta)) &= 1 \end{aligned}$$

Two intermediate quantities

Notation:

$$\text{imb}^+(\theta) := \max_{\mathbf{x}} \text{imb}(w(\mathbf{x}, \theta))$$

$$\text{imb}^-(\theta) := \min_{\mathbf{x}} \text{imb}(w(\mathbf{x}, \theta))$$

Questions

Let $\theta \in \mathcal{R}_k$.

- 1) Do we always have $\text{imb}^+(\theta) = k$?
- 2) When $\text{imb}^-(\theta) < k$, do we always have $\text{imb}^-(\theta) = 1$?
- 3) If no, do these quantities depend on the $\mathbb{Q}$ -dimension of θ ?

Results for $\text{imb}^+(\theta)$

Theorem 6 [Andrieu, V. 24+]

For every $\theta \in \mathcal{R}_k$, $\text{imb}^+(\theta) \geq k - 2$.

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Proposition 7 (optimality of Theorem 6) [Andrieu, V. 24+]

Let $d \geq 4$.

- i. For **every** $k \in \{3, \dots, d - 1\}$, there exists $\theta \in \text{int}(\mathcal{R}_k)$ s.t. $\text{imb}^+(\theta) = k - 1$.
- ii. For **some** $k \in \{3, \dots, d - 1\}$, there exists $\theta \in \text{int}(\mathcal{R}_k)$ s.t. $\text{imb}^+(\theta) = k - 2$.

Ex: For $\theta = (\frac{1}{d}, \dots, \frac{1}{d}) \in \text{int}(\mathcal{R}_3)$ we have $\text{imb}^+(\theta) = 1$.

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Ex: For $\theta = (\frac{1}{d}, \dots, \frac{1}{d}) \in \text{int}(\mathcal{R}_3)$ we have $\text{imb}^+(\theta) = 1$.

Open questions

- 1) Determine the values k for which there exists $\theta \in \mathcal{R}_k$ s.t. $\text{imb}^+(\theta) = k - 2$.
- 2) Characterize the momenta $\theta \in \mathcal{R}_k$ s.t. $\text{imb}^+(\theta) < k$.

Results for $\text{imb}^-(\theta)$

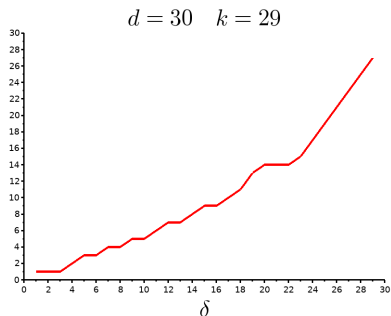
Theorem 8 [Andrieu, V. 24+]

Let $d \geq 4$, $k \in \{3, \dots, d-1\}$ and $\theta \in \mathcal{R}_k$. We denote by δ the $\mathbb{Q}$ -dim of θ . If $\delta < d$, then

$$\text{imb}^-(\theta) \geq \left\lceil \max\left(\frac{k-2}{q+1}, \frac{k-2-r}{q}\right) \right\rceil =: b(d, k, \delta)$$

where q and r are the quotient and remainder in the euclidean division of $2d-3-\delta$ by $\max(\delta-1, 1)$.

Ex: the lower bound for $d = 30$ and $k = 29$



Results for $\text{imb}^-(\theta)$

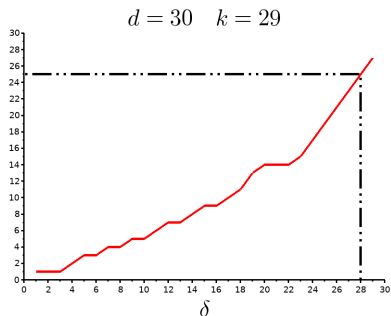
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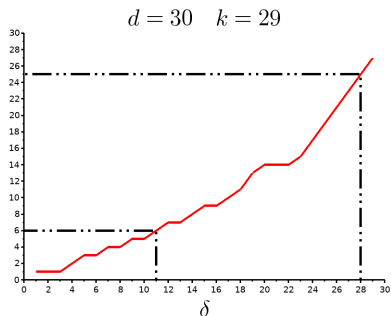
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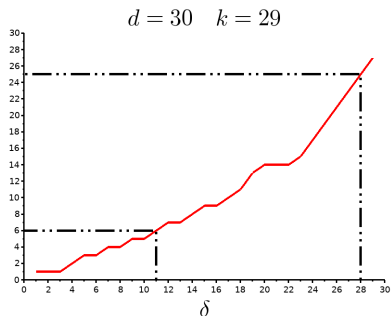
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Ex: the lower bound for $d=30$ and $k=29$

Questions

- 1) Does there really exist billiard words with a really low imbalance (compare to k)?
- 2) In other words, how far from reality/optimalty this lower bound is?



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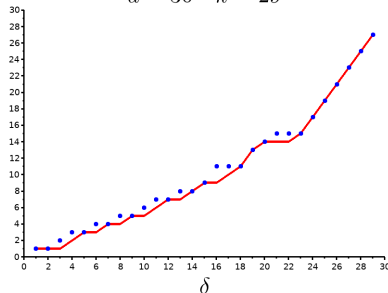
Ex: the lower bound for $d=30$ and $k=29$

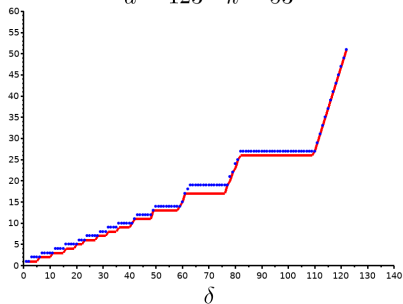
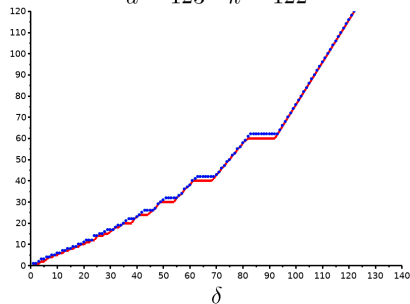
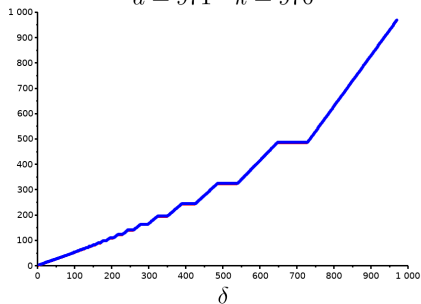
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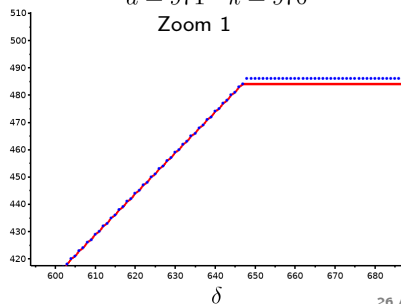
Empirical constructions suggest that the bound is good.

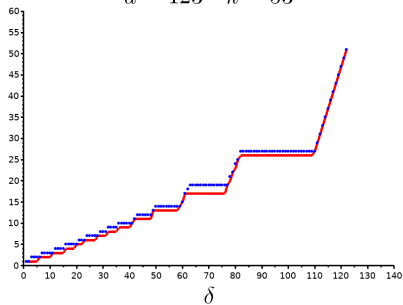
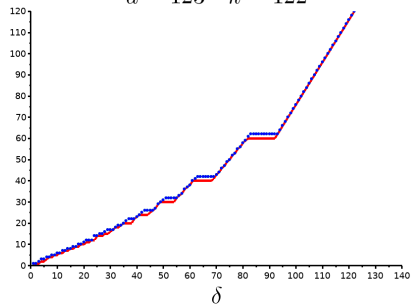
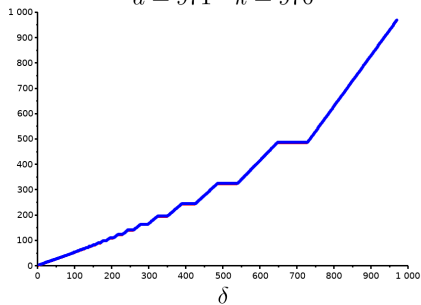
$d=30 \quad k=29$



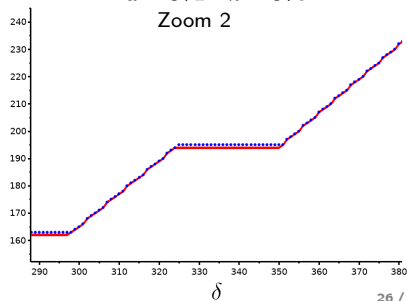
$d = 123 \quad k = 53$  $d = 123 \quad k = 122$  $d = 971 \quad k = 970$  $d = 971 \quad k = 970$

Zoom 1

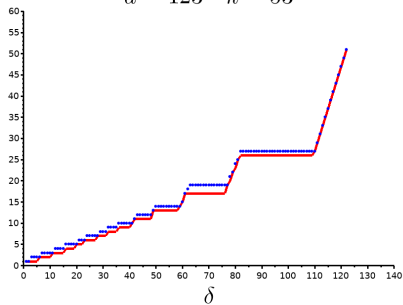


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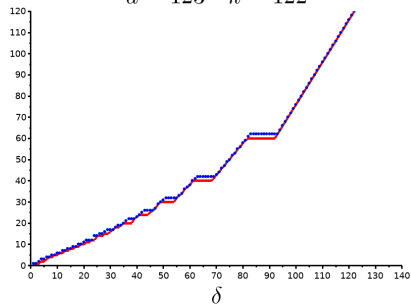
Zoom 2



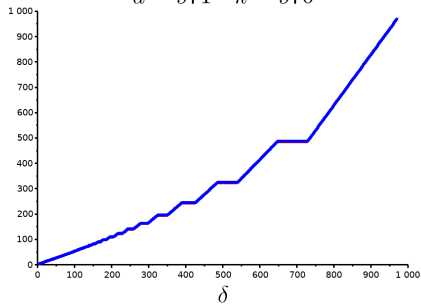
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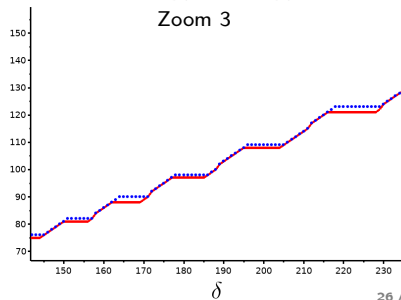


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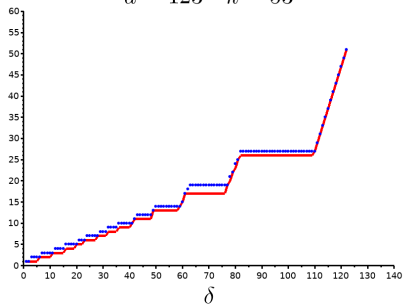


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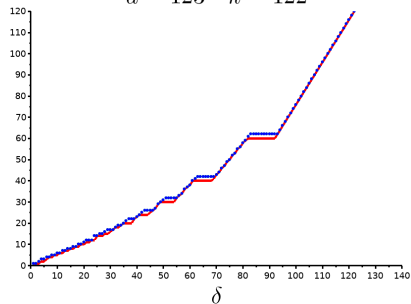
Zoom 3



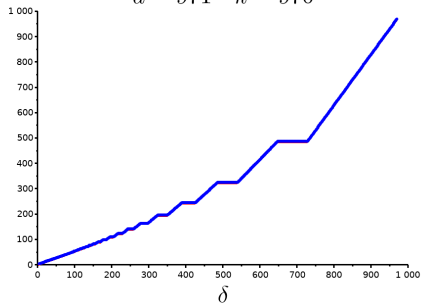
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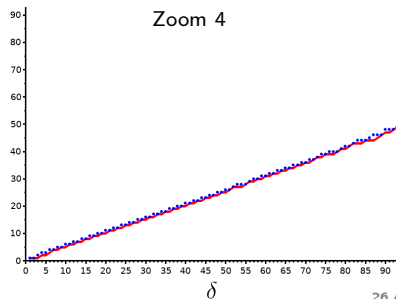


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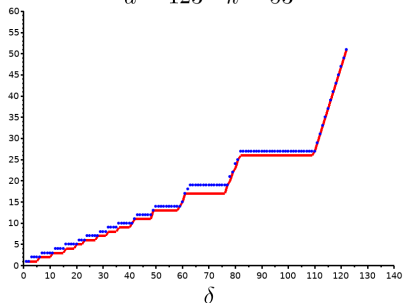


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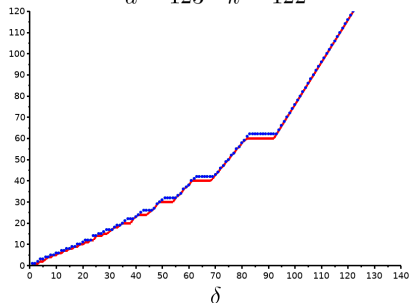
Zoom 4



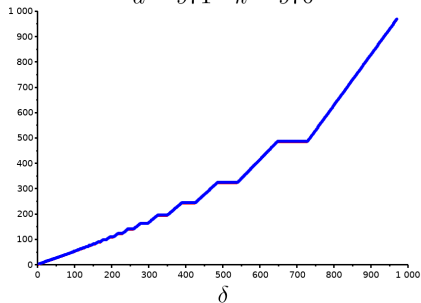
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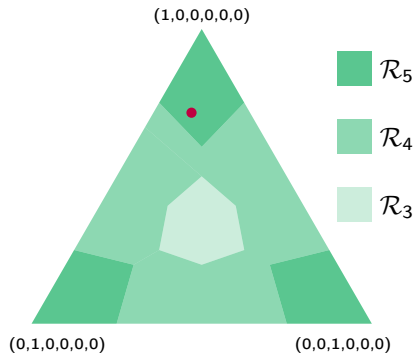
Open questions

- 1) Does there exist $C \in \mathbb{N}$ s.t. for every d, k, δ , there exists $\theta \in \mathcal{R}_k$ with $\mathbb{Q}\text{-dim} = \delta$ s.t.

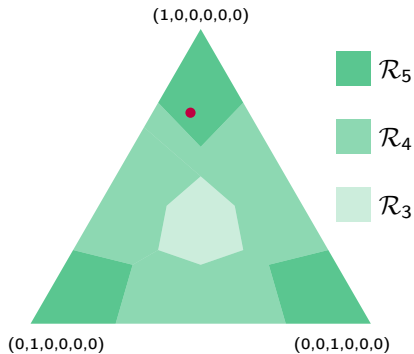
$$b(d, k, \delta) \leq \text{imb}^-(\theta) \stackrel{?}{\leq} b(d, k, \delta) + C ?$$

- 2) If yes, can we take $C = 2$?

Conclusion



Conclusion



Thank you!