

Cobham's theorem for Gaussian numerations

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Joint with

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Cobham's theorem

Numerations for the Gaussian integers

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Statement (again)

Ingredients

Proof Strategy

An obstacle to periodicity

“Proof”

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Theorem (Büchi, 1960)

*The set of integral powers of an integer $p \geq 2$ is **recognizable** in q -ary notation only if p and q are multiplicatively dependent, that is, only if $p^m q^n = 1$, for some $m, n \in \mathbb{Z}$.*

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Theorem (Cobham, 1969)

*A set $S \subset \mathbb{N}$ is recognizable in both p -ary and q -ary notation for some multiplicatively independent p and q if and only if it is **recognizable in r -ary notation for every $r \geq 1$.***

Theorem (Büchi, 1960)

The set of integral powers of an integer $p \geq 2$ is recognizable in q -ary notation only if p and q are multiplicatively dependent, that is, only if $p^m q^n = 1$, for some $m, n \in \mathbb{Z}$.

Theorem (Cobham, 1969)

A sequence $(a_n)_{n \in \mathbb{N}}$ is p - and q -automatic for some multiplicatively independent p and q if and only if it is eventually periodic.

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A more reasonable proof of Cobham's theorem

Cobham-type theorems for $\mathbb{Z}[i]$

- ▶ $\mathbb{Z}[i]$ is a Euclidean domain, with an algorithm to compute the quotient and the remainder,
- ▶ $\mathbb{Z}[i]$ has numeration systems (γ, D) , where any $z \in \mathbb{Z}[i]$ has a unique expansion as a linear combination of powers of γ , with “digits” in D .

Cobham-type theorems for $\mathbb{Z}[i]$

Theorem (Hansel & Safer, 2003)

Let p and q be two distinct positive integers and let $\alpha = -p + i$ and $\beta = -q + i$. Assuming that the four exponentials conjecture is true, if $S \subset \mathbb{Z}[i]$ is α - and β -recognizable, then it is syndetic.

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Theorem (Bustos-Gajardo, Fokkink & Y, 2026)

Let α and β be two multiplicatively independent Gaussian integers with $|\alpha|, |\beta| > 1$. If one of α or β is not the root of an integer, then a configuration $(a_z)_{z \in \mathbb{Z}[i]}$ is α - and β -automatic if and only if it is eventually periodic.

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Gaussian numerations

Let $\gamma \in \mathbb{Z}[\mathbf{i}]$ with $|\gamma| > 1$ and let $0 \in D \subset \mathbb{Z}[\mathbf{i}]$ be finite. For $w = w_{n-1} \dots w_0 \in D^*$ set

$$[w]_\gamma := \sum_{j=0}^{n-1} w_j \gamma^j.$$

$w \in D^*$ is a (γ, D) -*expansion* of $z \in \mathbb{Z}[\mathbf{i}]$ if $[w]_\gamma = z$.

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$D \subset \mathbb{Z}[i]$ is a *complete residue system* for $\mathbb{Z}[i] \pmod{\gamma}$ if every Gaussian z is congruent modulo γ to a unique element of D .

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$D \subset \mathbb{Z}[i]$ is a *complete residue system* for $\mathbb{Z}[i] \pmod{\gamma}$ if every Gaussian z is congruent modulo γ to a unique element of D .

(γ, D) is a *numeration system* if every $z \in \mathbb{Z}[i]$ has an expansion $z = [d_k \dots d_0]_\gamma$ with $d_j \in D$ and $d_k \neq 0$. If this expansion is unique, the numeration is *unambiguous*.

Abundance of numeration systems for $\mathbb{Z}[i]$

Theorem

Let $0 \neq \gamma = a + bi \in \mathbb{Z}[i]$, let $N := |\gamma|^2$ and let $\lambda = \gcd(a, b)$.

Suppose that $|\gamma| > 1$. Then:

1. $D(\gamma) = \left\{ p + iq : p = 0, 1, \dots, \frac{N}{\lambda} - 1, q = 0, 1, \dots, \lambda - 1 \right\}$ is
a complete residue system for $\mathbb{Z}[i] \bmod \gamma$.

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2. Let $0 \in D \subset \mathbb{Z}[i]$ be finite. If every $z \in \mathbb{Z}[i]$ has a (γ, D) -expansion, then (γ, D) is an unambiguous numeration system iff D is a complete residue system for $\mathbb{Z}[i] \bmod \gamma$.

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3. (K-S, 1975) Let $D = \{0, 1, \dots, N - 1\}$. Then (γ, D) is an unambiguous numeration system for $\mathbb{Z}[i]$ iff $\gamma = -n \pm i$.

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Theorem

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3. (K-S, 1975) Let $D = \{0, 1, \dots, N - 1\}$. Then (γ, D) is an unambiguous numeration system for $\mathbb{Z}[i]$ iff $\gamma = -n \pm i$.
4. (D-D-G, 1978) Let $|\gamma| \geq \sqrt{5}$ and let
$$D = \left\{ d \in \mathbb{Z}[i] : \operatorname{Re}\left(\frac{d}{\gamma}\right), \operatorname{Im}\left(\frac{d}{\gamma}\right) \in \left[-\frac{1}{2}, \frac{1}{2}\right) \right\}.$$

Then (γ, D) is an unambiguous numeration system.



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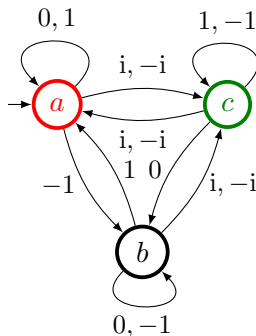
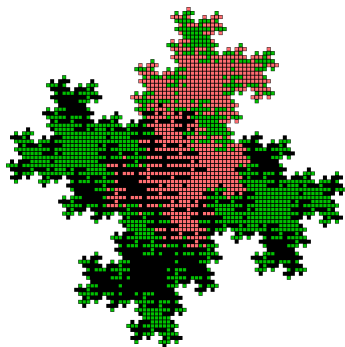
An obstacle to periodicity

“Proof”

Definition: (Consistently) automatic sequences

Suppose that (γ, D) is a numeration system.

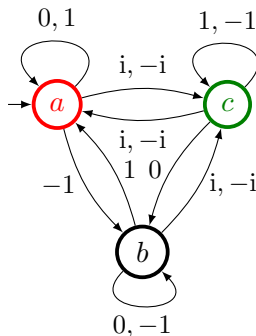
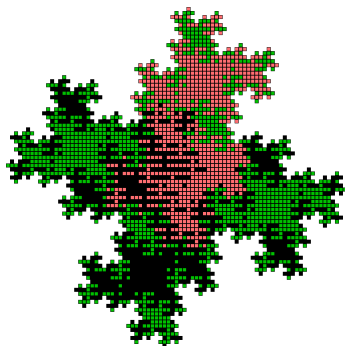
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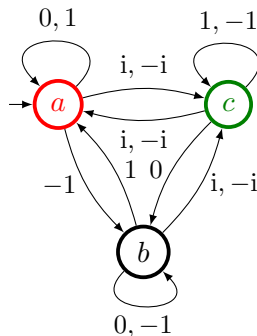
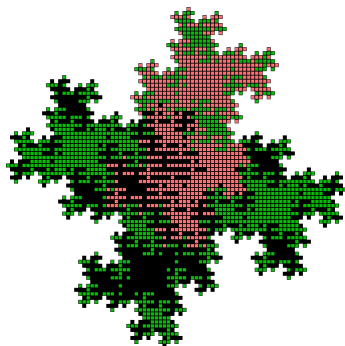


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$(a_z)_{z \in \mathbb{Z}[i]}$ is **(γ, D) -automatic** if there is a consistent $\mathcal{A}$ with $a_z = \tau(\delta(s_0, w))$ whenever $[w]_\gamma = z$.

Definition: Periodicity

Given two $\mathbb{Z}$ -linearly independent Gaussian integers p_1, p_2 , let

$$\langle p_1, p_2 \rangle = \{m_1 p_1 + m_2 p_2 : m_i \in \mathbb{Z}\}.$$

We write $z_1 \equiv z_2 \pmod{\langle p_1, p_2 \rangle}$ if $z_1 - z_2 \in \langle p_1, p_2 \rangle$.

We say that $(a_z)_{z \in \mathbb{Z}[i]}$ has *period* (p_1, p_2) in $U \subseteq \mathbb{Z}[i]$ if whenever $z_1, z_2 \in U$ and $z_1 \equiv z_2 \pmod{\langle p_1, p_2 \rangle}$, then $a_{z_1} = a_{z_2}$.

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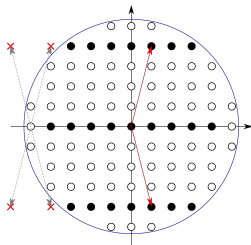
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We say that $(a_z)_{z \in \mathbb{Z}[i]}$ has **period (p_1, p_2)** in $U \subseteq \mathbb{Z}[i]$ if whenever $z_1, z_2 \in U$ and $z_1 \equiv z_2 \pmod{\langle p_1, p_2 \rangle}$, then $a_{z_1} = a_{z_2}$.

We say that $(a_z)_{z \in \mathbb{Z}[i]}$ has **step-period (p_1, p_2)** in U if whenever $z, z + p_i \in U$ then $a_z = a_{z+p_i}$.

Periodicity implies step-periodicity.

Step-periodicity in a reasonable U implies periodicity in a shrunk version of U .



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“Proof”

Theorem (Bustos-Gajardo, Fokkink & Y, 2026)

Let α and β be two multiplicatively independent Gaussian integers with $|\alpha|, |\beta| > 1$. If one of α or β is not the root of an integer, then a configuration $(a_z)_{z \in \mathbb{Z}[i]}$ is α - and β -automatic if and only if it is eventually periodic.

- ▶ Niven's theorem limits Gaussian integers that are roots of integers to have an argument that is a multiple of $\pi/4$.
- ▶ The set $\mathbb{N}$ is α -recognizable whenever α is a root of an integer.

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(1) Dirichlet approximation in $\mathbb{Z}[i]$

In his original proof, Cobham needed the fact that for multiplicatively independent $\alpha, \beta \geq 2$, the multiplicative group $\langle \alpha, \beta \rangle^\times = \{\alpha^m \beta^n : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}^+$.

Hansel and Safer showed that if α and β are multiplicatively independent Gaussians, then the four exponentials conjecture implies that $\langle \alpha, \beta \rangle^\times$ is dense in $\mathbb{C}$.

We only need that there are no isolated points in $\langle \alpha, \beta \rangle^\times$, a much weaker result.

Lemma

Let $\alpha, \beta \in \mathbb{Z}[i]$ be two multiplicatively independent Gaussian integers with $|\alpha|, |\beta| > 1$. Then 1 is not an isolated point in $\langle \alpha, \beta \rangle^\times$.

Thus given $\varepsilon > 0$ there exist $m, n > 0$ such that

$$|\alpha^m - \beta^n| < \varepsilon |\beta^n|.$$

(2) Consistent automata and “short expansions”

Krebs padded his digit sets to ensure short expansions.

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Lemma

Let (γ, D) be a Gaussian numeration system. Let $r \geq 2$ be chosen so that

$$D \subset D_\gamma := B(0, r|\gamma|) \cap \mathbb{Z}[i].$$

If $|z| \leq r|\gamma|^n$, then z has at least one (γ, D_γ) -expansion of length at most n .

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The next lemma ensures that consistent automata exist.

Lemma

Let (β, D) be a Gaussian numeration system and let $D \subset D'$. Then $(a_z)_{z \in \mathbb{Z}[i]}$ is (β, D) -automatic if and only if it is (β, D') -automatic, generated by a consistent automaton.

(3) Expanding step-periodic domains

Let

$$V^{or} := \{z \in V : B(z, r) \cap \mathbb{Z}[i] \subseteq V\}.$$

Lemma

Let $U \subseteq \mathbb{Z}[i]$ be a ball of radius R such that $(a_z)_{z \in \mathbb{Z}[i]}$ has *step-period* (p, q) in U . Then $(a_z)_{z \in \mathbb{Z}[i]}$ has *period* (p, q) in U^{or} for $r = |p| + |q|$.

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Lemma (“Fine-Wilf”)

Let $U, V \subseteq \mathbb{Z}[i]$ be two sets such that $(a_z)_{z \in \mathbb{Z}[i]}$ has a *step-period* (p, q) in U , and a *period* (p', q') in V . Let

$$\rho_1 = \max(|p|, |q|), \quad \rho_2 = \max(|p'|, |q'|), \quad R = \rho_1 + \rho_2.$$

If $U \cap V$ contains a ball of radius R , then $(a_z)_{z \in \mathbb{Z}[i]}$ has a *step-period* (p, q) in the set $U \cup V^{\circ \rho_1}$.

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Call a state s in $\mathcal{A}_\beta$ **juicy** if there are infinitely many z with expansions ending at s .

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Call a state s in $\mathcal{A}_\beta$ **juicy** if there are infinitely many z with expansions ending at s .

Almost all z have an expansion ending at a juicy state.

If β is not the root of an integer, then for each juicy s , **we will find linearly independent p_s, q_s , and $n \in \mathbb{N}$ such that $(a_z)_{z \in \mathbb{Z}[i]}$ has step-period (p_s, q_s) on a large ball centred at $(z + 1)\beta^n$ for any z which has a β -expansion ending at s .**

Henceforth: “Almost all” = “all but finitely many”.

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These balls will intersect nontrivially, and will cover of almost all $\mathbb{Z}[i]$, so we can use Fine-Wilf to conclude that the configuration is almost everywhere periodic.

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Creating non-collinear periods: an obstacle

Assume wlog that there is a juicy state s such that there is a cycle w rooted at s . Let w_1 be a word from the initial state s_0 to s .

For $j \in \mathbb{N}$, let $u_j = [w_1 w^j]_\beta$. If there exist k, ℓ such that $u_k, u_{k+\ell}, u_{k+2\ell}$ are collinear, then for some $m, \beta^m \in \mathbb{Z}$.

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Corollary (Creating linearly independent periods)

If β is not a root of an integer, then for any juicy state $s \in S_\beta$, there are non-collinear $x, y, z \in \mathbb{Z}[i]$ which have β -expansions ending at s , and α -expansions ending at the same state.

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1. Replace the numeration system (γ, D) by (γ, D_γ) , where $r \geq 2$ and

$$D \subset D_\gamma := B(0, r|\gamma|) \cap \mathbb{Z}[i].$$

If $z \in \mathbb{Z}[i]$ satisfies $|z| \leq r|\gamma|^n$, then z has at least one (γ, D_γ) -expansion of length at most n .

2. For any juicy state $s \in S_\beta$ there are non-collinear $x_s, y_s, z_s \in \mathbb{Z}[i]$ which have α -expansions ending at the same state, and β -expansions ending at s .
3. Choose m, n, K such that

$$|\alpha^m - \beta^n| \leq \frac{1}{K} |\beta^n|. \quad (\text{so } |\alpha^m| \approx |\beta^n|)$$

4. Define the linearly independent

$$p_s = (y_s - x_s)(\alpha^m - \beta^n), \quad q_s = (z_s - x_s)(\alpha^m - \beta^n).$$

Because of our choices we have

$$|p_s|, |q_s| \leq \frac{2}{K} |\beta^n|.$$

Final nail

Lemma

$(a_z)_{z \in \mathbb{Z}[i]}$ has step-period (p_s, q_s) on the ball $B((x+1)\beta^n, (1 - \frac{2}{K})\beta^n)$ if x has a β -expansion ending at s .

Proof Let $z, z + p_s \in B(\beta^n, (1 - \frac{2}{K})\beta^n)$.

Then:

1. z has a β -expansion of length at most n ,
2. $z - y_s(\alpha^m - \beta^n)$ has an α -expansion of length at most m ,
3. $z + p_s$ has a β -expansion of length at most n .

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We have

$$\begin{aligned} a_x \beta^n + z &= a_{y_s} \beta^n + z \\ &= a_{y_s} \alpha^m + z - y_s(\alpha^m - \beta^n) \\ &= a_{x_s} \alpha^m + z - y_s(\alpha^m - \beta^n) \\ &= a_{x_s} \beta^n + z + p_s \\ &= a_x \beta^n + z + p_s. \end{aligned}$$

What happens in other Euclidean domains?