

Palindromic length in free groups, Catalan expressions, and new risks for mathematicians

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Palindromic length

The palindromic length $pl(u)$ is the minimal number of concatenated palindromes needed to express the word u .

$$abbaba = (abba) (b) (a) = (a) (bb) (aba)$$

$$pl(abbaba) = 3$$

Do we consider a free semigroup or a free group? Or maybe something else?

Semigroup or group?

$$abca$$

In the semigroup: $pl(abca) = 4$.

In the free group: $pl(abca) = 3$, since

$$abca = (aba) (a^l a^l) (aca).$$

The difference can be unbounded [Saarela, 17]:

$$AbcA^p = (a_1 \cdots a_n)bc(a_n \cdots a_1) = (AbA^p)(A^l p A^l)(AcA^p).$$

$2n + 2$ vs. 3 palindromes

In the semigroup

- Studied since [F., Puzynina, Zamboni, 2013];
- can be computed in linear time [Borozdin, Kosolobov, Rubinchik, Shur, 2017];
- is unbounded for factors of aperiodic infinite words [Rukavicka, 2026].

In the free group

- Studied since [Bardakov, Shpilrain, Tolstykh, 2005];
- unbounded on $aba^2b^2 \cdots a^nb^n$;
- [Saarela 2017]: important examples.

Story of the problem

- [Bardakov, Shpilrain, Tolstykh, 2005]: Is there an algorithm to compute the group palindromic length?
- Around 2018: I learn about the question in a private communication with Valery Bardakov.
- 2019: a characterization of words of palindromic length 2 and a conjecture for the pal. length 3 are given in a Master thesis of my student M. Bou-Laouz.

Small palindromic lengths in free groups and word equations with antimorphisms

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Abstract

The palindromic length of a finite word w is defined as the minimal number of palindromes such that their product is w . Clearly, this function may take different values depending on if we consider w as an element a free semigroup or of a free group: for example, in the free semigroup, the palindromic length of $abca$ is 4 (here every letter is a palindrome), and in the free group, it is 3 since $abca = (aba)(a^{-1}a^{-1})(aca)$.

In free semigroups, the palindromic length can clearly be computed, and there are fast algorithms for that. In free groups, the question is trickier. In this paper, we characterize words in the free group whose palindromic length is 2 and 3.

Conjecture deep inside

So, with possible forms of words of palindromic lengths 2 and 3, and with a particular conjecture on the palindromic length 4, I also stated the following:

This fact and the first values 1, 1, 2 and perhaps 5 of the number of possible expressions for $k = 1, 2, 3, 4$, suggest that perhaps the reduced forms may somehow be related to Dick paths, and their numbers are Catalan numbers.

Chronology

- My preprint is online since December 12 2025;
- January 2026: Volker Diekert offers a collaboration with him, Alex Thumm and Silas Natterer; I accept.
- May-June 2026: we exchange drafts and ideas by mail;
- August 2026: Volker and Alex come to Marseille, and we prove several beautiful results;
- September 11: I announce this talk online:

Upcoming talks		Past talks	
Your time	Speaker	Title	Saved
Tue Sep 29 15:00	Anna Frid	Palindromic length in free groups and Catalan expressions	☒
Tue Oct 13 15:00	Vanessa Kosoy	TBA	☒

Palindromic Length in Free Groups: Reflections, Noncrossing Matchings, and Catalan Forms

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Abstract

Let $F = F(X)$ be a free group of finite rank, with palindromic length taken with respect to the fixed basis X . We embed F as the index-two subgroup of the universal Coxeter group $W = F \rtimes_{\theta} \langle t \mid t^2 = 1 \rangle$, where $\theta(x) = x^{-1}$ for $x \in X$, and prove

$$\text{pl}(g) = \min\{\ell_T(g), \ell_T(gt)\}.$$

State of art

- We have more results than one can extract from the title, and more than what I will announce today;
- Our preprint is in preparation;
- But it seems that even announcing titles can be risky in the age of AI (if we consider it as a risk).

Initial result

Theorem 1

A word is a concatenation of 2 palindromes in the free group if and only if its reduced form looks like

$$xp_1p_2x^\iota$$

for some (possibly empty) palindromes p_1, p_2 and a (possibly empty) word x .

$$\implies : xp_1p_2x^\iota = (xp_1x^\rho) (x^\iota p_2x^\iota).$$

Proof (one of the cases)

$\Leftarrow$: suppose that $(xsx^\rho) \cdot (yty^\rho) = xy^\rho$

$$sx^\rho = t^\ell y^\ell$$

$$s = t^\ell p_1 p_2 \quad \text{or} \quad t^\ell = s p_1 p_2$$

$$p_1 p_2 x^\rho = y^\ell$$

$$y = x^{\ell\rho} p_2^\ell p_1^\ell$$

$$y^\rho = p_1^\ell p_2^\ell x^\ell$$

$$\boxed{xy^\rho = x p_1^\ell p_2^\ell x^\ell}$$

Three palindromes

Theorem 2 [F., 2025]

A word is a concatenation of 3 palindromes in the free group if and only if its reduced form looks like

$$x_1 x_2 p_1 p_2 x_2^t p_3 x_1^p$$

or

$$x_1 p_1 x_2 p_2 p_3 x_2^t x_1^p$$

for some (possibly empty) palindromes p_1, p_2, p_3 and a (possibly empty) words x_1, x_2 .

The proof is technical with many cases and subcases.

Catalan expressions

A *P-expression* is a word over $\{p, (,)\}$ with correctly placed parentheses.

$$(((p(p))pp))()p.$$

A *P-expression* is *minimal* if all parentheses are needed (no $((E))$, no $()$, no (p)).

$$(((p(p))pp))()p \mapsto ((pp)pp)p.$$

A *P-expression* is a *Catalan* one if it is minimal and contains a maximal number of parentheses:

$$((pp)pp)p \mapsto (((pp)(pp))p) \text{ or } (((((pp)p)p)p)p).$$

Catalan expressions

- All Catalan expressions can be generated by the context-free grammar

$$E \rightarrow (EE)|p .$$

- There are C_n Catalan expressions with $n + 1$ palindromes,

like $((pp)p)$ and $(p(pp))$ for $n = 2$

and $((pp)p)p, ((pp)(pp)), ((p(pp))p), (p(p(pp))), (p((pp)p))$ for $n = 3$.

Decompositions

Let u be a P -expression.

A u -decomposition of a word $w \in A^*$ is a decomposition of w to reduced factors according to the letters of u as follows:

- every factor corresponding a letter p is a palindrome;
- the factors corresponding to a pair of parentheses are of the form x and x^α , where

$$\alpha = \begin{cases} \iota, & \text{if the number of palindromes inside is even,} \\ \rho, & \text{if it is odd.} \end{cases}$$

$$(((pp)p)p) \mapsto x_1 x_2 x_3 p_1 p_2 x_3^\iota p_3 x_2^\rho p_4 x_1^\iota.$$

General result

Theorem

A word is a concatenation of at most n palindromes in the free group if and only if its reduced form admits a u -decomposition for a P -expression u with n letters p .

Example

$$(((pp)p)p) \mapsto x_1 x_2 x_3 p_1 p_2 x_3^t p_3 x_2^t p_4 x_1^t.$$

$$ababbbabbabbab^t aab^t abbba^t = a \cdot b \cdot \cdot abbba \cdot bbabb \cdot \cdot ab^t aab^t a \cdot b \cdot bb \cdot a^t$$

Idea of the proof

We start from a non-reduced concatenation of n palindromes.

It admits a $\underbrace{pp \cdots p}_n$ -decomposition.

Now we consider partial reductions which touch only two symbols of the P -expression and look how the expression changes:

- ① $pp \rightarrow (pp)$;
- ② $(pE), (Ep), (E)p, p(E) \rightarrow (p(E)) \mid ((E)p)$;
- ③ $() \rightarrow ()$;
- ④ $(E_1)(E_2), ((E_1)E_2), (E_1(E_2)) \rightarrow ((E_1)(E_2))$.

Erasing parentheses

Lemma

Let $w \in A^$ admit a u -decomposition to a P -expression u . Then either w or a reduction of w admits a decomposition to a minimal P -expression with the same number of palindromes.*

- ① $((E)) = xyEy^\alpha x^\alpha = zEz^\alpha \rightarrow (E) ;$
- ② $(p) = xpx^p \rightarrow p ;$
- ③ $() = xx^\ell \rightarrow \varepsilon .$

Adding parentheses

Now we can add parentheses to a minimal P -expression to get a Catalan one. The images of new parentheses in the reduced word are empty.

$$a \cdot b \cdot \cdot abbb a \cdot bbabb \cdot \cdot ab^t aab^t a \cdot b \cdot bb \cdot a^t \rightarrow ((ppp)p) \rightarrow (((pp)p)p)$$

So, the theorem holds.

Polynomial time

Theorem

The free group palindromic length of a given word can be found in polynomial time.

Idea of the proof: Given a reduced word w of length n , we consider all possible decompositions

$$w = xuvx^\alpha,$$

where α is ι or ρ .

There are $O(n^2)$ such decompositions.

We have $pl(w) \leq k$ if and only if there exist u and v with $pl(u) = m$ and $pl(v) = k - m$ for some m (with the correct correct α).

+ dynamic programming to compute pl of each factor only once.



Conclusion

- So, we found a beautiful combinatorial description of words with free group palindromic length $\leq n$: they have the structure of a Catalan expression with $n - 1$ pairs of parentheses.
- My coauthors have also other results, more sophisticated and general.
- But in the era of AI, we cannot announce them or even give keywords before the preprint is online.